Yocum.—Teaching of Addition and Subtraction

An Inquiry into the Teaching of Addition and Subtraction

THESIS

Presented to the Faculty of Philosophy of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

BY

ALBERT DUNCAN YOCUM

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INTRODUCTORY.

In the following discussion it is assumed that whatever the numerical knowledge which children should ultimately possess, and whatever the time when their formal instruction in number should begin, they must at the very outset become thoroughly familiar with the fundamental facts of addition and subtraction; that painstaking inquiry should be made into the numerical content of their minds on their first entering school; that so far as possible this content should become the basis of elementary arithmetical instruction; that aside from the order in which the fundamental number-facts can be most readily mastered by each individual, there is a psychological order in which they should be taught to pupils studying en masse—*i. e.,* that there is an order in which they can be most readily mastered by the great majority of individuals who are under common instruction; that similarly there is a psychological method by which they should be taught; and that such order and such method may be suggested by *a priori* discussion and determined by practical investigation.

This inquiry therefore begins with a résumé of the results of all investigations into the numerical content of children's minds on entering school which were accessible to the writer, supplemented by the data obtained in a somewhat restricted investigation of his own; attempts to prove that there is no necessary antagonism between logical order and psychological order; seeks to indicate what on *a priori* grounds, should be the psychological order and method of teaching the fundamental facts of addition and subtraction; and finally reports the results of the use of the order and method thus indicated, in the schools of a large manufacturing town.

While the conclusions resulting from this inquiry are purely tentative, and may of course be disproved by broader investigation, it must be none the less remembered that they are based upon facts and have been successfully subjected to the test of impartial practice.

Throughout the entire discussion effort has been made to avoid all difficulties not necessarily involved in the simple problem which it seeks to solve. It cheerfully leaves to the mathematical philosopher the controversy over the number concept, taking it for granted, however, that such concept will have either as an important factor or a necessary condition, an intelligent knowledge of the decimal system, as well as of the fact that each number in the scale is greater by one than that which immediately precedes it.

With equal firmness it turns aside from seductive speculations concerning the origin of number and its historic development, assuming that even should the Culture Epoch Theory be regarded as demonstrated, the study of the child would be more likely to throw light upon the history of number, than that of barbarous dialects and ancient texts upon the order in which number should be taught the child.

What number shall be taught to children, whether it shall be taught them on their first entering school, what proportion the time devoted to it shall bear to that given to other branches of the curriculum,—these and kindred questions of pedagogic importance it fails to discuss, not through a minimizing of their importance, but from the fact that they are judged worthy of special investigation.

Certain fundamental sums and differences must be mastered by the children. Intelligently derived, they must be mechanically used in numerical operation. If this investigation serves to indicate an order and a method of instruction which will bring about their readier mastery by the majority of the pupils when taught in common, it will accomplish the end for which it was undertaken,—

not, it is hoped, with the result that more number-work will be done in a given school year, but that less time need be devoted to the number-work at present demanded by school curricula, whatever that time may be.

It is only proper that these introductory remarks should conclude with formal acknowledgment of the practical assistance afforded by Dr. Lightner Witmer, Assistant Professor of Psychology, and the suggestive discussions which characterize his seminaries. To him and to Dr. Brumbaugh, Professor of Pedagogy, is due in no small measure such success as may have attended the inquiry.

CHAPTER I.—THE NUMERICAL CONTENT OF CHILDREN'S MINDS UPON THEIR FIRST ENTERING SCHOOL.

From 1869, when the Berlin Pedagogical Union initiated the first general inquiry into the ideational range (Vorstellungskreiz) of children entering school,[1] such tests have been common in the pedagogic centres of Germany. Nor has the least service which Dr. Hall has done to education in America been the impetus given to similar investigation through the forceful demonstration of its value and suggestive directions as to its method, contained in his now classical report on the tests in Boston and Kansas City schools.[2] Numerous, however, as these tests have been, they have shed but little light upon the numerical knowledge possessed by children at the beginning of their school career. Among the 10,000 first tested in Berlin, 74.35 per cent were found to be familiar with the number 2; 73.99 per cent, with 3, and 72.65 per cent, with 4—and even this sparse data loses much of its value, when it is known that some of the returns from which it was obtained were made after the children had been several weeks in school. These are the only facts concerning children's numerical knowledge which the tests sought to obtain, and yet inquiry was made into their familiarity with almost a hundred concepts. When one takes into account the proportionate time given to number-work in the curriculum of the elementary school, together with the fact that a main object in these and succeeding tests was to determine what knowledge the beginner could be certainly assumed to possess, the neglect to further test his numerical knowledge can be understood only by assuming that investigators have taken it for granted—perhaps quite unconsciously—that the child enters school prac-

tically ignorant of even the simplest facts of number. Dr. Lange fails to touch the subject of number at all in his test of 500 beginners at Plauen, and 300 in the surrounding country.[3] Dr. Hall himself after a searching inquiry as to how many among 200 Boston children fail to have any proper concept of a hundred simple and familiar things, only adds to the numerical data, resulting from the Berlin test, that 8 per cent of his subjects did not know the "number-name" 3; 17 per cent, 4, and 29.5 per cent, 5.[4] This he does because "number cannot be developed to any practical extent without knowledge of the number-name,"[5]—that is, he assumes a minimum of knowledge necessary to the teaching of number, then seeks to discover how many children do not possess it.

Indeed all these inquiries into the general content of children's minds have necessarily been negative in their results. The knowledge which children may possess is so extensive and so various, that a given test, even though limited to that part of the possible content, likely to be common to a majority of individuals, while showing much that the individual does not know, can show but little of that which he does. On the contrary, a special inquiry into the numerical content of children's minds being very limited in its extent, can be quite positive in its results.

Another characteristic of Dr. Hall's test was that his 200 children were carefully selected, being limited to those of "average capacity," and care being taken to "avoid schools where the children came from homes representing extremes of either culture or ignorance."[6] This usual scientific precaution must of course be omitted in a special inquiry, which seeks among other things to determine the numerical knowledge which can be counted upon as common to the great majority of beginners, without previous knowledge as to their capacity, and with no regard to their environment. This objection does not hold against the results of Hartmann's test at Annaberg on the 660 boys and the 652 girls, between the ages of six and

three-fourth and five and three-fourth years, who began school from 1881-84. Of these, 69 per cent of the boys and 62 per cent of the girls could count from 1 to 10.[7] In Superintendent Greenwood's tests in Kansas City, identical with those of Dr. Hall and reported by him, the "number-names" were omitted,—probably owing to the fact that at the time of the test the children had been some seven months in school.

It would seem then that the teacher,—discouraged by the difficulties encountered in imparting the fundamental numerical facts, has led the scientist to assume that the beginner knows little or nothing of number. More than this, it is already being urged on what may be or may not be good physiological and psychological grounds, that he *ought* to know nothing of it, and that he should not be taught it at the time when he first enters school. "The child," it is said, "is too young for abstractions; he is still in the imaginative epoch of his cultural development; certain nerve fibers are not yet medullated; nature must not be forced,"—an argument which the assumption of his numerical ignorance tends somewhat to reinforce.

But since aside from the sparse data already summarized, the teacher has only dogma upon which to depend, it becomes necessary for a stable pedagogy to determine by adequate investigation the numerical content of children's minds on their first entering school.

The following test was made primarily to determine in so far as was possible, what certain individual children knew on entering the schools of a particular locality, and secondarily as a step in the more general determination of what numerical knowledge the great majority of American children may be assumed to have in common on entering the public school. For obvious reasons it was not made by the regular teachers, but by experienced teachers' assistants, who had been carefully instructed together in a uniform mode of procedure. These assistants were seven young women—one for each school,— accustomed

to the management of children and selected with special reference to their fitness for the task. The results obtained by them were checked by the regular teachers as they began the work of the year.

The test was made on the first day of school, in order that its results might be reached before any of its subjects had been in charge of a teacher, or had been given any systematic instruction. Fifty boys and fifty girls were subjected to it. The average age of the former was 5.64 years; of the latter 5.48 years. The highest age given was eight years, two months. Only thirty children were older than six years. That a number of children admitted were below school age there can be little doubt, as many hardworking parents look upon the school as a day-nursery, and feel that a misrepresentation is excusable which will relieve them from serious care, and at the same time admit their children a little earlier to educational advantages which they must relinquish all too soon. The test was made in a manufacturing town of from 10,000 to 12,000 inhabitants, comprising an unusually small proportion of professional men, but by way of compensation, very few foreigners. The children tested, coming from every section of the town, are in all probability fairly representative of the great mass of native-born beginners in the country as a whole.

Great care was taken to so put the questions that the results may be universally valid. Where a fact was put concretely reference was strictly limited to such objects as all American school children are likely to have counted —marbles for the boys, candies and pennies for both boys and girls. A possible exception is to be found in the use of apple, top, and orange in Questions XII to XV. The second part of Question XIX is theoretically unfortunate owing to the tendency of children to answer "yes," when they think that an affirmative will please. Practically, however, this tendency would be hard to trace,— very few children answering the question at all. Ques-

tions VII and VIII should have been supplemented elsewhere by others in which the facts were not arranged in series, in order to determine how far regular arrangement was helpful to those who gave them correctly. The later experience of teachers, however, showed conclusively to what extent such was the case. As a possible source of serious error any embarrassment that could be detected was noted. Most children, however, were natural in their manner, and seemed to respond freely to questions which they were able to answer.

Suggestion was from the very nature of the questions impossible. Prompting was permitted in counting, but note was made wherever it occurred. No conditions of the experiment, therefore, seem such as to seriously modify its results.

As the recording of the answers given was so planned as to involve a minimum amount of writing, each investigator filled out her own blanks. Although designed as they are to be within certain limits fairly exhaustive, the questions seem quite numerous, it will be seen on investigation, that it was unnecessary to ask all of every individual. The average time taken for each pupil was about ten minutes. The following is a copy of the blank.

MATHEMATICAL CONTENT OF CHILDREN'S MINDS ON THEIR FIRST ENTERING SCHOOL.

I. Age............ Years............ Months.............

II. Boy or girl?

III. American or Foreign?

IV. Manner: Natural or embarrassed? (An X placed above a word or figure will indicate it as the answer; above a fact in number, that it has been correctly given; above a question, that it has been correctly answered.)

V. Can he count? Yes or no. To what number without objects? 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. (Cross out any numbers omitted by him, or on which—while hesitating—he was prompted.)

VI. To what number can he count actual objects? 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. (Cross out as above.)

VII. Which of the following sums can he give correctly: ("One and one more?" "Two and one more?" etc., is the form in which these sums should be required; first, *abstractly*—correct replies being denoted by the usual X; second, *concretely*—correct replies being denoted by an O placed after the X). 1 & 1, 2 & 1, 3 & 1, 4 & 1, 5 & 1? (Only when the preceding sums are correctly given should the following be required): 6 & 1, 7 & 1, 8 & 1, 9 & 1, 10 & 1, 11 & 1, 12 & 1, 13 & 1, 14 & 1, 15 & 1, 16 & 1, 17 & 1, 18 & 1, 19 & 1, 20 & 1?

VIII. Which of the following differences can he give correctly? ("If you have one and I take away one, how many have you left?" etc., is the form in which these differences should be required, both abstractly and concretely as above). 1 — 1, 2 — 1, 3 — 1, 4 — 1, 5 — 1, 6 — 1? (Only when the preceding differences are correctly given should the following be required): 7 — 1, 8 — 1, 9 — 1, 10 — 1, 11 — 1, 12 — 1, 13 — 1, 14 — 1, 15 — 1, 16 — 1, 17 — 1, 18 — 1, 19 — 1, 20 — 1?

IX. Particular Number Associations.—Does he know without counting how many hands he has? Yes, or...... How many fingers on one hand? Yes, or...... The number of legs a horse has? Yes, or ...... (If his answer to one of the questions under IX is a wrong number, put it in the blank; if so to one under X, put it above each successive number in place of the X.)

X. Number Perception too Rapid for Conscious Counting.—Can he instantly tell the number of splints successively shown him in groups of 2, 4, 6, 3 and 5, held distinctly apart in one hand? The total number when 2 are thus held in *each* hand? When 3 are so held?

XI. Can he correctly give the following facts? (First abstractly, then concretely, and indicated as above): 1 — 1, 2 — 2, 3 — 3, 4 — 4, 5 — 5, 6 — 6? 2 & 2, 2 & 3, 3 & 2, 4 & 2, 2 & 4, 3 & 3? 3 — 2, 4 — 2, 4 — 3, 5 — 2, 5 — 3, 5 — 4, 6 — 2, 6 — 4, 6 — 3, 6 — 5?

XII. If one apple costs 1c., what will two apples cost?........

XIII. If a ball costs 2c., what will three balls cost?........

XIV. If a top costs 2c., how many can I buy for 4c.?........

XV. If an orange costs 3c. how many can I buy for 6c.?........

XVI. 2 ones? 2 twos? 2 threes? 3 ones? 3 twos?

XVII. How many twos in 4? In 6? How many threes in 6?

XVIII. You have three sticks and I have two sticks—Which has more? ....How many more?........ Which less?........ How many less?........

XIX. Into how many parts do you cut an apple to divide it in half? ....Should those parts be of the same size?........ To divide it into thirds?........ Into quarters?........

Name of Pupil..................................
Name of Teacher................................

PERCENTAGES FOR FIFTY BOYS AND FIFTY GIRLS.

	Boys.		*Girls.*	
	No.	Per Cent.	No.	Per Cent.
V. Can Not Count at All...............	4	8	2	4
Without Objects	6	12	10	20
With Objects	2	4	2	4
Can Count With and Without Objects..	38	76	36	72
Without Objects From 2 to 5..	39	78	38	76
5 to 10..	39	78	33	66
10 to 20..	31	62	26	52
VI. With Objects From ....2 to 5..	43	86	46	92
5 to 10..	37	74	36	72
10 to 20..	24	48	28	56
Can Count Higher Without Objects...	10	20	10	20
With Objects	16	32	19	38
VII. Know Without Objects..... 1 + 1..	14	28	19	38
2 + 1..	20	40	19	38
to 10 + 1..	11	22	13	26
20 + 1..	5	10	8	16
With Objects........ 1 + 1..	31	62	35	70
2 + 1..	32	64	35	70
to 10 + 1..	10	20	17	34
20 + 1..	4	8	7	14
VIII. Know Without Objects.... 2 — 1..	11	22	17	34
3 — 1..	11	22	14	28
to 10 — 1..	4	8	9	18
With Objects........2 — 1..	18	36	25	50
3 — 1..	17	34	22	44
to 10 — 1..	3	6	11	22
IX. Know number Hands...............	41	82	41	82
Fingers	23	46	29	58
Horse's Legs	39	78	38	76

	Boys. No.	Boys. Per Cent.	Girls. No.	Girls. Per Cent.
X. Recognize 2	37	74	35	70
3	27	54	30	60
4	23	46	23	46
5	11	22	11	22
6	7	14	6	12
2 Two Groups	21	42	19	38
2 Three Groups	6	12	4	8
XI. Know Without Objects 1 — 1	6	12	16	32
2 — 2	8	16	14	28
to 6 — 6	9	18	14	28
With Objects 1 — 1	13	26	21	42
2 — 2	17	34	21	42
to 6 — 6	14	28	21	42
Without Objects 2 + 2	12	24	6	12
2 + 3	12	24	5	10
to 3 + 3	8	16	3	6
With Objects 2 + 2	10	20	14	28
2 + 3	6	12	18	36
to 3 + 3	5	10	6	12
Without Objects 3 — 2	11	22	6	12
4 — 2	5	10	4	8
to 6 — 5	4	8	4	8
With Objects 3 — 2	10	20	11	22
4 — 2	7	14	6	12
to 6 — 5	4	8	6	12
XII. Know 1 × 2 Concretely	28	56	29	58
XIII. Know 2 × 3 Concretely	1	2	3	6
XIV. Know 2's in 4 Concretely	4	8	7	14
XV. Know 3's in 6 Concretely	3	6	3	6
XVI. Know 1 × 2 Abstractly	7	14	8	16
2 × 2 Abstractly	4	8	3	6
3 × 2 Abstractly	2	4	2	4
1 × 3 Abstractly	3	6	3	6
2 × 3 Abstractly	1	2	2	4
XVII. Know 2's in 4 Abstractly	3	6	2	4
2's in 6 Abstractly	3	6	1	2
3's in 6 Abstractly	2	4	1	2
XVIII. Understand Term "More"	41	82	36	72
"How Many More"	8	16	9	18
Term "Less"	14	28	11	22
"How Many Less"	2	4	3	6

	Boys. No.	Boys. Per Cent.	Girls. No.	Girls. Per Cent.
XIX. Know That There Are Two Halves in Whole	27	54	18	36
That These Parts Should Be Equal	18	36	11	22
That There Are Three Thirds in Whole	5	10	2	4
That There Are Four Quarters in Whole	5	10	2	4

Summary of Results.

While the results obtained from the testing of so small a number of pupils are by no means conclusive, they are none the less significant. No generalizations not purely tentative can be based upon them alone; scientifically, they are merely suggestive, or at most but a step in the induction of positive knowledge; but pedagogically they are final, in that definitely showing the numerical knowledge of certain individual children who are to be taught together, they certainly indicate the numerical known to which the unknown facts of number are to be joined and through which they are to be apperceived.

The first significant fact to note is that the girls knew quite as much of number as the boys,—if not somewhat more. If they are at a disadvantage later on in the arithmetical course, it is not due to a poor start as compared with that of their brothers.

Again, as was to be expected,—there was noticeable a great variation in the character of the knowledge displayed by various individuals,—ranging from the almost absolute ignorance of the three or four who knew the least, to the remarkable readiness with which two or three gave the most difficult facts. In the case of the majority, however, the knowledge of individual facts was so scattered and unsystematic as to compel the teachers to ignore it, if they taught in the usual logical order. There were few children who did not have a considerable knowledge

of number, but in scarcely any two cases was that knowledge the same. Eighteen per cent of those tested knew abstractly that 2 and 2 are 4, and 17 per cent that 2 and 3 are 5, while concretely 24 per cent knew these same facts. Eleven boys and six girls knew 3—2 as an abstract fact, while ten boys and seven girls knew it in the concrete. But all those who knew it in the concrete did not know it in the abstract, nor did the same boys and girls who knew the differences always know the sums. With any method of teaching which took up in regular order the combinations and separations of the numbers from 2 to 20, a classification which would from the start group together in separate divisions pupils having approximately the same degree of development was plainly impossible.

All this seems at first glance to justify the assumption of practical ignorance to which reference has already been made, and the consequent practice—far too common—by which with few exceptions beginners are grouped together at the start, and only separated into divisions—if separated at all—when they are found to widely vary in the facility with which they acquire the elementary facts.

The fact, however, none the less remains, that the majority of the children, thus tested on entering school, displayed a considerable knowledge of number. There were only four boys and two girls who could not count at all, approximately three-fourths of the whole number could count both with and without objects. Of those who could not, six boys and ten girls who could count actual objects, could not count abstractly, while two boys and two girls who could count abstractly, could not count actual objects, —that is to say, with but 4 per cent of the whole number of pupils, was the counting merely mechanical. Ten boys and ten girls could count higher without objects than with; sixteen boys and nineteen girls higher with than without. More than half of the pupils, however, could count equally high in the abstract and in the concrete. As

full comprehension of the counting process can call for nothing in addition to this but the generalization that each number is one greater than that immediately below it in the scale,[13] it is surprising to learn that so many of the children possessed the knowledge necessary to it. While but 24 per cent of the whole number knew that 2 cents and 2 cents are 4 cents, 49 per cent knew even abstractly that 2 and 1 are 3, while 62 per cent knew that 2 cents and 1 cent are 3 cents, thus beginning a generalization, which 24 per cent had carried to 10 and 1 abstractly and 27 per cent concretely, and respectively 13 per cent and 11 per cent to 20 and 1. Seventy-seven per cent knew that 2 is more than 1, and 17 per cent answered correctly what must have been the unfamiliar question, "How many more?" It was not remarkable that but 25 per cent could tell whether 1 was "less" than 2 or 2 than 1, and that but three or four knew how many less. Fifty-four per cent of the boys and 36 per cent of the girls knew that a thing to be divided in half must be cut into two parts, and 36 per cent of the boys and 22 per cent of the girls understood that these parts should be equal. Ten per cent of the boys and 4 per cent of the girls knew the number of parts into which a whole must be cut to divide it into thirds or into fourths when called quarters.

Although 18 per cent of the beginners could not tell how many hands they had, 48 per cent the number of fingers on one hand, and 23 per cent the number of a horse's legs, the customary assumption that such lack of knowledge is indicative of an ignorance of number was not proven. On the contrary many thus ignorant knew as much as those who were not, their ignorance being due to the fact that their attention had not been directed to the numbering of the particular concrete things in question. Many adults would be found on investigation unable to give the number of ribs in the human frame, or of legs normally possessed by a fly.

It is significant, too, that 72 per cent of the children

recognized the number 2 with counting made practically impossible; 57 per cent, the number 3; 46 per cent, 4; 22 per cent, 5; and 13 per cent, 6. When two splints were held in each hand, 40 per cent recognized the total as 4, and when three were so held, 10 per cent the total as 6. With but very few exceptions, all children who knew the number of hands they had, could recognize 2 without counting, and all who knew a horse had four legs could so recognize 4—something which scarce a child without that knowledge could do. It is significant in this connection that while 77 per cent knew the number of legs a horse has, though horses' legs are in pairs, only 24 per cent knew that 2 and 2 are 4, and 7 per cent, two 2's.

Incidentally acquired though this knowledge is, scant in the case of the few, but quite considerable in that of the majority, if repeated tests in various localities confirm the foregoing results, it will at least be certainly demonstrated that the great majority of children know something of number on entering school. Whether they have enough in common for the public school teacher to be able to utilize it, will be discussed later.

Comments Upon the Foregoing Results.

Whatever the facts that may be ultimately associated with a given digit, however various the numerical relations in which it plays a part, primarily it seems to stand for aggregation. The child knows that 1 and 1 are 2 before he learns that two 1's are 2; that 5 and 1 are 6 before he discovers or is led to discover that three 2's are 6. He asks for "another" or "one more" rather than for two times as many as he had before. He learns to count before he masters the elementary products. Multiplication as taught him in the school may never be to him "a short method of addition," but nevertheless were he left to himself, 8 would be 4 and 4, before it would be 4 taken two times, or multiplied by 2. It has been left for the philos-

opher to assert the fact that number is ratio. The child counts before he measures. To him addition is counting on, subtraction is counting off. Counting, then, is the fundamental process.[8] It begins—in so far as it is not merely mechanical—with the child's plea for "just one more," his perception of one thing and another one, not, of course, with the abstraction 1 and 1.[9] Sixty-six per cent of the 100 children tested on entering school knew that 1 cent and another cent are 2 cents; only 33 per cent that 1 and 1 are 2. Half, however, of those who knew this primal fact of number at all, had made the generalization necessary to apply it to any one thing and another,—a fact which is of significance in a later connection. While only four boys and two girls were unable to count at all, thirty-four boys and girls did not know that 1 more than 1 familiar thing, or 1 familiar object and another,—are 2. Hence it is plainly obvious that many children who are reported as being able to count were completely ignorant of the fact that each successive digit is one more than the one which immediately precedes it. It would be erroneous, however, to assume that counting thus unintelligent, is a purely mechanical repetition. Only 16 per cent of the whole number of children were unable to count actual objects, and thus to definitely number a group of 1's, of whose number as compared with other groups they had no knowledge. For example, the great majority of beginners were at least able to determine that there were four objects in a given group, though ignorant of the fact that their number was 1 more than that of a group of 3 or 1 less than that of a group of 5, while a very considerable number had made the generalization necessary to intelligent counting. These results tend to demonstrate that upon entering school, a few children will be found who cannot count at all (in this case 6 per cent);[10] a few whose counting is merely mechanical repetition originating in the series concept,[8] conventionalized by imitation, and stimulated by a love of rhythm (16 per cent);[11] a

large number who having taken the next step can by counting determine within narrow limits the number of objects in a given group though ignorant that each digit which they name is greater by 1 than that immediately preceding it (34 per cent);[12] a still larger number (44 per cent), comprising almost half of the children tested,—who have at least begun to count intelligently.[13]

Counting, then, it would seem, is at first to the great majority of children a rhythmical series of names with little or no meaning, sooner or later correctly applied to particular groups of objects, and gradually perceived as a series of number-names each one of which may be applied to a group containing one more object than does that to which the number-name last preceding it may be applied. To some, counting may never be a mechanical process. They may successively add by 1's, may know that 1 and 1 are 2, 2 and 1, 3, etc., before they repeat 1, 2, 3, as a series. Indeed this addition of successive 1's and the mechanical counting may go on side by side, without the child's perceiving any connection between them.

The fact that "counting on" or "forwards" is so popular a process with children is likely to make it a matter of imitation before it becomes intelligent; not so, however, with "counting off." Very few of the children tested could count backwards from 10 to 1, yet 28 per cent knew in the abstract that 2 less 1 are 1, 25 per cent that 3 less 1 are 2, and 13 per cent the most of the remaining facts to 10 less 1, while in the concrete the percentages were severally 43 per cent, 39 per cent and 14 per cent. The fact that few children knew these differences in any systematic order strongly indicates that they were mastered as separate and distinct facts. They did not seem to know that 3 less 1 are 2, because 2 is next below 3 in the digit series. Unlike the sums formed by adding 1, there was nothing so far as could be seen—to differentiate the facts in which 1 was subtrahend from those in which the subtrahend was 2 or 3. It is evident, then, that the

only way in which these children became familiar with certain differences was the process by which they came to know such of the elementary sums as were not obtained by adding 1. Just as they counted out the number of objects to be added and then counted out their sum, so they must have counted out the number of objects in the minuend, the number that was taken away and the number remaining. They were almost without exception, ignorant of the term "less;" subtraction was to them a "taking away" with a certain remainder, not the *lessening* of a number by a certain amount. The number remaining does not seem to have been inferred from the number to which the subtrahend is added to obtain the minuend. That is, they did not know that 3 less 2 is 1, because they already knew that 2 and 1 are 3, but because they met with it in their experience as a separate and distinct fact. Twenty-three per cent abstractly and 35 per cent concretely seem to have made the generalization that when a number is taken from itself the remainder is "nothing."

However few the disconnected facts of number which a child masters as distinct operations, and which repeated in his experience with different objects become abstractions, it is probable that they would be fewer still did he have to laboriously count out *every* number which they involve. Three-fourths of the children tested could recognize 2 without noticeable counting; more than half, 3; and a little less than half, 4. Twenty-two per cent and 13 per cent respectively thus recognized 5 and 6. As the objects used in this test were splints held fan-like and displayed but for an instant, the resulting forms were hardly those that the child would already have associated with particular number-names. This being so, while I fully agree with Dr. Phillips that much which passes as number recognition "is only the recognition of an individual form,"[14] in the present case the children seem to have recognized *number*. It may be more than a mere coincidence, however, that with but very few exceptions, all

children who knew the number of hands they had, could recognize 2, and that all who knew a horse has four legs, could recognize 4,—something which scarce a child without that knowledge could do. Two objects can not be so placed that they will not be in a straight line, three and four objects are usually so grouped that they may respectively form vertices for triangles and quadrilaterals. The relative position of the horse's legs, human hands or the feet of a three-legged stool is not different from that of the corresponding number of splints.

In the absence of recognition by the children of what is thus in effect an imaginary *general* form,—far-fetched though its assumption may seem,—whether we grant the truth of Külpe's hypothesis of a direct recognition, dependent upon the "effectiveness" of a percept for central excitation and the "mood" which it induces,[15] or assume that the recognition is here due to the comparison of the present percept with a memory image in part identical,—an adequate explanation of this instantaneous number recognition is exceedingly difficult. In the first place, whatever may be true in the case of the adult, in that of the child just entering school, there is little likelihood of unconscious counting, and no possibility whatever of either conscious or unconscious estimation. Few of the children tested were orally rapid counters, and the great majority in counting objects assisted discrimination by pointing to each successive one or by handling it. It is altogether unlikely that a "mental counting" of which there was no external sign—would be rapid, if indeed in the case of the majority—it could even be possible with any degree of accuracy.

Tests in which the pupils would be led to so count the numbers certainly recognized would be indeterminate, in that in the absence of introspection the observer would not determine whether the result was reached through counting or not, while if they were found able to "mentally" count higher numbers, the results would not

necessarily apply to the lesser numbers directly recognized.

Unconscious counting being thus improbable, unconscious estimation was plainly impossible, where a majority of the subjects were ignorant of the necessary facts. The grouping of objects instead of helping to a more accurate judgment, resulted in the reverse,—only 40 per cent of the pupils being able to recognize two 2 groups as 4, and 10 per cent two 3 groups as 6. The separation into two groups only made it more difficult to perceive the objects as one whole.

While it may be well to recall in this connection the famous experiments in which young children through the display of a number of large and irregular objects side by side with a larger number of smaller objects, have been deceived into pronouncing the former number the greater, they merely serve to demonstrate that children, like adults, may be led to form a wrong judgment of comparative number.

In conclusion, then, we may state as true of the particular children tested and tentatively assume as true of American children in general:

1. That the majority of children have some knowledge of number on first entering school.

2. That to them counting is the initial numerical process, and the ability to count, the only numerical knowledge which they have in common.

3. That they have little or no knowledge of ratio.

4. That to them the numbers 1, 2, 3 and 4 have become concepts which in most cases can be instantaneously and correctly applied whenever the corresponding groups of objects are distinctly seen.

CHAPTER II.—AN A PRIORI DETERMINATION OF THE PSYCHOLOGICAL ORDER OF TEACHING NUMBER.

I. DISCUSSION OF LOGICAL AND OF PSYCHOLOGICAL ORDER.

As has been already asserted, it is impossible to utilize the numerical knowledge possessed by children on entering school, if number is to be taught them in the usual logical order,—that is, in the order which exhausts the combinations and separations of each successive number from 1 to 20 before the teaching of any fact connected with those above it. The question therefore naturally arises, is this logical order the only order in which number can be taught, and if not, is it—all things considered—the order in which the fundamental facts of addition and subtraction can be most readily mastered by beginners? There is a tendency among disciples of "the new education" to take for granted, an antagonism between "logical" and "psychological" order. The knowledge of the primary school, it is assumed, must not be systematic knowledge, because the order in which facts are best grouped by the adult mind is not that in which they can be most readily apperceived by the mind of the child. Now it has by no means been demonstrated that knowledge logically arranged can not be more readily comprehended by the beginner than can any aggregation of isolated facts, however simple those facts may be. It is characteristic of a logical order that facts having something in common are grouped together. A logical classification, basing itself on an element common to a large

number of things, will separate out from the mass two or more groups which will together contain all things having the common element, but which are differentiated from each other in that the things in each group have an additional common element, which those in the others have not. Similarly each group may then be separated into additional groups, the partial identity of the things in each sub-group growing greater and their number less, until it is impossible for partial identity to be thus increased. Facts are taught in a logical order, when beginning with the things having least in common, instruction successively passes to those which have most, or when beginning with those which have the most in common, it similarly passes to those which have least. Teaching which thus goes from the general to the particular is called deductive; that which taking the opposite course, passes from the particular to the general, is known as inductive. As the things included in a given class may be differently grouped accordingly as this or that element common to many of them, is made the basis of classification, the same things may usually be classified in many different ways, with the result that the young may be taught both deductively and inductively in more than one logical order. Now aside from such general conditions as attention and fatigue, nothing is more necessary to the processes involved in apperception and recall, than such arrangement of individual facts as makes more readily perceivable the presence of some common element.

It is only when the common element is too insignificant for children to readily perceive it, or when the things to which it is common are too complex or too various for it to be readily perceived, that facts logically grouped can not be readily mastered. The psychological order of teaching a number of particular facts, is that through which a given individual or group of individuals can most readily approximate their full apperception and prompt recall. While the psychological order thus concerns itself with

the succession in which facts are presented to the individual mind, and the logical with the manner in which they are related to each other, it is readily to be seen that the former is conditioned by the latter.

Not only, therefore, is there no necessary antagonism between the logical and the psychological orders, but it may be seriously questioned whether to the vast majority of individuals any order of teaching the facts of the school curriculum is fully psychological without being logical. If the facts to be imparted to a given individual have no element in common, that order of teaching will be psychological, which so presents them that each separate fact will precede all others more difficult for that individual to perceive. If on the contrary the facts to be thus imparted have anything in common, which mastered in one will materially aid in the mastery of all, that order of teaching will be psychological, which groups together the facts possessing this common element, provided that it is readily perceivable to the individual taught. That such a grouping is logical is evident *per se*. Since the great majority of facts included within the school curriculum, have more or less in common with other facts, it necessarily follows that wherever possible, they must be presented in a logical order whose common element readily perceived by the majority of pupils, in the mastery of one or more individual facts, will insure the ready mastery of the remaining facts with which they are grouped. It will be convenient to designate such a common element as an "effective" element.

The logical order will therefore be psychological, when its common element is one thus effective, and readily perceivable to the individuals taught. Other things being equal, the larger the number of the facts included in its several groups, the more psychological will an order be, for the larger the number of facts in the groups, the smaller the number of groups, and hence the readier the mastery of the whole body of facts. Moreover it is plain

that the common element should not only be effective, but that it should be that particular effective element which is the basis of scientific classification, wherever such an element is readily perceivable by the individuals taught.

To assert that without exception all psychological order of grouping the facts included in the school curriculum must be logical order would be to assume not only that each of these facts has in common with other facts more or less that is effective, but that every fact has among the various elements thus common and effective at least one which through psychological instruction can be readily perceived by the great majority of individuals taught. While this assumption is quite unnecessary to the present argument, experience has abundantly demonstrated that as a rule such an element can be found in the various facts which go to make up the common school curriculum.

If a given logical order based upon an essential common element seems too difficult for the child, before condemning it as unpsychological, it would be well to know whether the attempt to impart the facts so grouped has been made in accordance with a psychological method of instruction. (See Chapter III.) And even should this prove to be the case, the failure of the child to comprehend them is no indication that logical order in general is unpsychological, but rather that in a particular logical order, the common element can not be readily perceived by the child mind,— the right course in such an event being to seek to arrange the same facts in an order no less logical, but in which the essential common element can be more readily perceived.

The discussion thus far has made clear the conditions necessary to the psychological sub-grouping of the facts included in any general category.

In each psychological group,—

1. The common element must be an effective one; one whose perception in the mastery of one fact in the group will insure the ready mastery of every other fact.

2. It must be the most effective one which through psychological instruction can be readily perceived by the great majority of particular individuals to be taught, unless some other effective element thus readily perceivable is the basis of the accepted scientific classification.

3. It must be common to more facts than is any other readily perceivable and equally effective.

All the facts of a general category that must be mastered by the child, having been arranged so far as possible in psychological groups, it is evident that the psychological order of teaching these groups will depend upon the relative effectiveness of their common elements, the relative readiness with which their common elements are perceived by particular individuals taught, and the relative number of facts which they include. When, as is usually the case, mastery of the effective element of one group is a necessary condition to that of the effective element of another, or when, more rarely, the effectiveness, perceivability and inclusiveness of the elements common to the various groups increase or diminish together, psychological order is readily determinable. But when, as may possibly occur, among a number of unrelated groups, the common element of one has relatively the highest effectiveness, of another the readiest perceivability, and of a third the greatest inclusiveness, it is not so easy to determine which of the three should be taught first. Each group, it may be, can be readily mastered by the child, but should they vary considerably in the readiness with which they can be mastered, and not too widely in the number of facts which they include, it is probably more psychological to teach first the one which, whether through the higher effectiveness of its common element or through the greater readiness with which that element is perceivable, can be mastered *most* readily. If on the contrary one should include many more facts than the others, being effective and readily mastered, though not *most* readily

mastered, it should be taught first, tending as it does to the readier mastery of the whole.

Where, as will usually be the case with psychological groups, the groups themselves have elements in common they should be grouped together in their turn, under the same conditions as the single facts which they themselves include, it being remembered that here, the readier mastery of some one group must be subordinated to the ready mastery of the whole collection of groups of which it is a part.

It is fortunate that this general discussion of psychological order has not involved the determination of what knowledge is of most worth. Whether or not physiology should be taught before number or æsthetics before reading, it is not for psychology to determine. But whatever the sequence of the general branches of study—utilitarian, ethical, or historic, there will be few among the facts selected as necessary to the child, that can not be included in some psychological group. Relative usefulness, character building, or culture epoch may demand that some individual fact shall be taught before the psychological group of which it would otherwise form a part, or that some logical group shall be taught before the effective element common to its members can be readily mastered by the child, but after all such demands have been met, there is an order in which the facts thus included from year to year in the school curriculum can be most readily mastered by a majority of the pupils taught, and this order is psychological.

However great the variation in our courses of study, the facts which are taught in school are all included in some general classification. They are grouped together into branches of study in each school curriculum, and taught for the most part with the aid of separate text-books. Even in the primary grades they are grouped as number-work, nature-study, and the like. The common element upon which this general classification is based

soon comes to be recognized—though perhaps somewhat vaguely—by even the youngest pupil. He is not able to tell what it is, unless the teacher has pointed it out or denoted it by some suggestive term, but he will not confuse a number-fact with his reading or a sentence with his number,—unless "correlation" has quite unified his mental content. At the point, however, when on successive subdivision, the common element, however effective, is not readily perceivable or where being readily perceivable instruction does not lead the learner to perceive it, logical order becomes of no avail, except in so far as it prevents the instructor from omitting any of the necessary facts. To the learner each fact stands by itself, unrelated to other facts, and is perceived by him without being well apperceived. It was this misuse of logical order that characterized the so-called formal instruction which here and there still exists in American schools. To the influence of German pedagogy and more latterly to the popularization of psycho-genesis—especially in the form of "child study," has been largely due the reaction against formal instruction, which is now almost everywhere triumphant. Unfortunately, however, iconoclasm here as elsewhere the companion of reform, has shattered the old idol into a thousand fragments, and thereby too often substituted for logical order in elementary instruction, the study of isolated facts.

II. THE PSYCHOLOGICAL ORDER OF TEACHING THE FUNDAMENTAL SUMS AND DIFFERENCES.

(I) The Reaction Against Formalism in the Teaching of Elementary Number.

The changes which have taken place in the study of arithmetic during the past fifty years well illustrate this swing of the pedagogic pendulum. But a generation or so ago formalism reigned supreme. Beginners were taught

notation and numeration to billions, before they began addition; the addition of columns long and broad, before they studied subtraction, and so on and on through a logical order which they were rarely led to comprehend, counting on fingers, learning by rote, and working by rule, they plodded from multiplication table to the rule of three. No one will claim that to the majority of those taught such teaching was psychological, but notwithstanding its theoretic faults, the reformer is occasionally non-plussed by the undeniable fact that it turned out a host of strong mathematicians.

The reason is not far to seek. All that was lacking, where the element common to each group was an effective common element, was that it should be comprehended by the individuals taught. And many a plodder did comprehend even as he labored, while many another came with the aid of a maturer mind, to ultimately comprehend what had been mechanically but permanently drilled into his memory. To such the old formal instruction was either psychological or became so. Not so, however, with the great majority, stupefied by a logic they could not understand, and arrested in mental development by what was to them for the most part the mechanical memorizing of disconnected facts. With the more or less individual instruction of the old district school, the formal grouping of the facts of number was undoubtedly helpful to the few. But as no educational plan, through its helpfulness to a minority can find justification for its application to all, educational leaders—largely influenced by the German reformers—began to protest against the mechanical mastery of fact and rule, and to urge in its place the teaching of "intellectual arithmetic." In school journal and teachers' meeting they persistently insisted that the pupil should not recite rules "by heart without thinking or considering thereon," that in the solution of problems he should examine the "question," and not his stock of rules; that, in fact, there should be no rules at all, but in their stead

a few general processes which would furnish the key to all solutions. Thus arithmetical analysis and mental arithmetic came to find a place in the curriculum,—substituting for rigidity of rule a no less rigidity in demonstration, but necessitating the teaching of the subjectively simple, before that of the subjectively complex. The triumph of reform, even yet by no means complete, was not won without a struggle. The embattled host of tradition long continued to hold its own against the champions of reason. The educational literature of the period is rich in material for a history yet to be written, of this stage in the old warfare between "Trojan" and "Greek." But while here and there may be found a locality where every rule has been hurled from its sacred niche, there has hardly yet resulted, as feared by a grandiloquent conservatism, a generation of pupils, who independent alike of teacher and text-book, "never soar by faith to regions of inconceivable glory, but ever plod on by the dim light of reason."[16]

The only innovation that need be here discussed is the change which took place in the order of teaching the elementary facts of number. Largely through the influence of Colburn and Emerson, the fundamental facts of addition, subtraction, multiplication and division began to be taught before the study of numeration and notation, or operation with numbers greater than 10 or 20. Later came the introduction of the Grube method,[17] involving another order no less logical than the old but which has for its end the exhaustion of the "simpler" of the fundamental facts before the study of those objectively more complex. No effort was made to so group the facts that the mastery of one might involve the ready mastery of many. On the contrary ready mastery was to come through the method by which each individual fact was to be imparted. Even to-day while the Grube *method* is rapidly being displaced by what are perhaps its betters, the Grube *order* still reigns supreme in American

schools.[18] Is this order psychological? If not, what is the psychological order of teaching the elementary sums and differences?

(II) Description of the Various Logical Orders Possible in Teaching the Elementary Sums and Differences.

The terms "sum" and "difference" are here used to denote not merely the results of certain numerical operations, but in their popular signification, as denoting the operations themselves, including their results. For example, $3 + 2 = 5$ is called a sum, and $5 - 3 = 2$, a difference. An "elementary" sum is one formed by the addition of one digit to another; a "secondary" sum, one in which one or both of the numbers added are greater than nine. The "inversion" of a sum only differs from the sum itself, in that the number to which addition is made has changed place with the number to be added. An "elementary" difference is one formed by the subtraction of one digit from another; a "secondary" difference, one in which either subtrahend or remainder is greater than nine. An "alternation" is a difference in which the remainder has changed places with the subtrahend. All sums and differences, whether elementary or secondary, whose mastery is necessary to ready operation, may be regarded as "fundamental." These fundamental sums and differences together with the fundamental products, quotients, etc., are commonly known as "the number facts."

As the secondary facts are readily derivable from those that are elementary, and the identity of result in sum and inversion, or the interchange of result in difference and alternation, are—as will be presently demonstrated—readily perceivable by the great majority of children, it is unnecessary for the present to consider the arrangement of all fundamental facts that are not elementary, or of elementary facts that are inversions or alternations.

Again, while it has not been taken for granted that the elementary differences should not be taught contemporaneously with the elementary sums, it will better serve the purposes of the discussion to fix the psychological order for the teaching of the latter, before considering whether the former should be taught thus contemporaneously, at a later period, or not at all.

Can the elementary sums and differences be grouped in more than one logical order? If so, what are these logical orders, and which among them is the psychological order for the majority of children on first entering the lowest school grade? It is not until this latter question has been fully determined by a strict application of the test already laid down that we can judge as to how far we can utilize the knowledge of number which children possess on first entering school.

In building up a group of different elementary sums it is evident that but one term can remain constant. If the result—the "sum" in the stricter sense of the term—is not varied, both the number to be added and that to which addition is to be made, become variables; if either of the latter is kept constant, the number added to it or to which it is added, and their result, must be variables. Moreover, it is evident that if a system of grouping is to include all of the elementary sums, it must involve the addition of each digit to every other digit,—though not necessarily in the usual digit order. Hence only three distinct sets of groups of elementary sums are possible: (1) Those in which the "result" is constant; (2) those in which the digit to which addition is to be made, is constant, and (3) those in which the number to be added is constant. In each set every digit must be the constant in a group containing all the elementary sums, which can be formed by the variation of the other two terms. If the order of teaching the elementary sums is to be psychological in the highest degree it must be determined:

1. Whether one system of grouping is psychological

throughout as compared with either of the others, or whether psychological order necessitates the use of groups from more than one system:

2. The psychological order of the groups selected.

3. The psychological order of the facts within each group.

For convenience in the discussion, the forty-five elementary sums not inversions are given below, grouped according to each system and following the order of the digits.

TABLE OF THE ELEMENTARY SUMS ARRANGED UNDER EACH OF THE THREE LOGICAL SYSTEMS OF GROUPING.

A. The Grouping Together of the Sums Obtained by the Addition of the Several Digits in Order to Each Successive Digit.

1. 1 & 1, 1 & 2, 1 & 3, 1 & 4, 1 & 5, 1 & 6, 1 & 7, 1 & 8, 1 & 9;
2. 2 & 2, 2 & 3, 2 & 4, 2 & 5, 2 & 6, 2 & 7, 2 & 8, 2 & 9;
3. 3 & 3, 3 & 4, 3 & 5, 3 & 6, 3 & 7, 3 & 8, 3 & 9;
4. 4 & 4, 4 & 5, 4 & 6, 4 & 7, 4 & 8, 4 & 9;
5. 5 & 5, 5 & 6, 5 & 7, 5 & 8, 5 & 9;
6. 6 & 6, 6 & 7, 6 & 8, 6 & 9;
7. 7 & 7, 7 & 8, 7 & 9;
8. 8 & 8, 8 & 9;
9. 9 & 9.

B. The Grouping Under Each Successive Digit of the Sums Whose Results Are Equal to It.

1. 1 & 1;
2. 2 & 1;
3. 3 & 1, 2 & 2;
4. 4 & 1, 3 & 2;
5. 5 & 1, 4 & 2, 3 & 3;
6. 6 & 1, 5 & 2, 4 & 3;
7. 7 & 1, 6 & 2, 5 & 3, 4 & 4;
8. 8 & 1, 7 & 2, 6 & 3, 5 & 4;
9. 9 & 1, 8 & 2, 7 & 3, 6 & 4, 5 & 5;
10. 9 & 2, 8 & 3, 7 & 4, 6 & 5;
11. 9 & 3, 8 & 4, 7 & 5, 6 & 6;
12. 9 & 4, 8 & 5, 7 & 6;
13. 9 & 5, 8 & 6, 7 & 7;
14. 9 & 6, 8 & 7;
15. 9 & 7, 8 & 8;
16. 9 & 8;
17. 9 & 9.

C. The Grouping Together of the Sums Obtained by the Successive Addition of Each Digit, First to Itself, Then to Each Inferior Digit.

1. 1 & 1, 2 & 1, 3 & 1, 4 & 1, 5 & 1, 6 & 1, 7 & 1, 8 & 1, 9 & 1;

2. (1) 2&2, 4&2, 6&2, 8&2; 5. (1) 5&5; 6. (6) —— 8. (3) —— 9. (4) —
(2) 3&2, 5&2, 7&2, 9&2; (2) 6&5; 7. (1) 7&7; (4) —— (5) —

3. (1) 3&3, 6&3, 9&3;
   (2) 4&3, 7&3;
   (3) 5&3, 8&3;
4. (1) 4&4, 8&4;
   (2) 5&4, 9&4;
   (3) 6&4;
   (4) 7&4;

   (3) 7&5;
   (4) 8&5;
   (5) 9&5;
6. (1) 6&6;
   (2) 7&6;
   (3) 8&6;
   (4) 9&6;
   (5) ——

   (2) 8&7;
   (3) 9&7;
   (4) ——
   (5) ——
   (6) ——
   (7) ——
8. (1) 8&8;
   (2) 9&8;

   (5) ——
   (6) ——
   (7) ——
   (8) ——
9. (1) 9&9;
   (2) ——
   (3) ——

   (6) —
   (7) —
   (8) —
   (9) —

(III) Comparison of the Three Systems with a View to the Determination of the Psychological Order.

Is one system of grouping more psychological throughout than is either of the others?

To determine this, the usual test must be applied to the various groups resulting from the completed development of each system. Have the groups of any one system, as compared with those of the other two, common elements:

1. More effective?
2. More readily perceivable by those to be taught?
3. Common to a greater number of the elementary sums?

It must be steadily held in mind that among these tests that of effective identity is of the highest comparative importance,—that, if the mastery of one individual of the group is not the key to the mastery of all, the grouping can be psychological only in so far as it insures the teaching of facts subjectively simple before those relatively complex.

Certain numbers are to be firmly associated together as individual number facts. If any considerable number of these facts can be so grouped together that the mastery of one involves the mastery of all, their partial identity may be regarded as an effective identity. The group is psychological, however, only when this effective identity is readily perceivable by a majority of the individuals who are to thus firmly associate the digits together as facts. The more effective this identity, the more readily it is per-

ceivable, and the greater the number of facts which possess it, the more psychological the resulting group.

1. *Why the Grouping Together of the Sums Obtained by the Addition of the Several Digits in Order to Each Successive Digit is Not the Psychological System of Grouping.*

This grouping which results from the addition of all the digits to each digit as a constant, prevailed in American schools from the time of Colburn until the Grube method became popular. It is found for example in such well-known text-books as those of Emerson (1838), Greenleaf (1851), and Davies (1862), as well as in many others. The constant addend to which addition is made and the fact that when the sums are taken in the order of the digits, each result is greater by one than that immediately preceding it, are the elements common to all the facts of a given group, not considered in relation to other groups. There is nothing, however, in the fact that the pupil has added one number to 6, to aid him in giving the result obtained by the addition of another number to 6, unless he successively gives all the facts which, when taken in order, intervene between them. Thus, knowing that 6 and 2 are 8, he can know that 6 and 5 are 11, by repeating to himself, 6 and 2 are 8, 6 and 3 are 9, 6 and 4 are 10, 6 and 5 are 11,—an operation which was usually performed with the aid of finger-counting.

But this knowledge is not due to the grouping. The result may be just as readily obtained for the isolated fact. The method employed is merely the counting out of the sum, either mentally or with the aid of objects. It is evident then that the elements common to the various groups of this system, considered within themselves, are not effective. When, however, the group formed by the addition of the digits to 5 has been thoroughly mastered before the study of the 6 group begins, the facts included in the latter have an effective common element,—in that

any fact in the 6 group will have a result greater by one than that of the corresponding fact in the 5 group. If 5 and 5 are 10, 6 and 5, or 1 and 5 and 5 will be 11. This, however, is not so readily perceivable by the child as is the fact that if 5 and 5 are 10, 5 and 5 and 1, or 5 and 6 will be 11—which is, as will be seen, all that it is necessary for him to perceive in the third system of grouping. In the former case he must be led to see that the addition of 1 to the 5, is the addition of 1 to the 5 and 5; in the latter, that the addition of the 1 to the 5 and 5 involves the addition of the 1 to the 5. As the result can be obtained inductively only by first adding the 1 to the 5 and 5, or the 5 and 5 to the 1, it is evident that the latter method of derivation will be more readily mastered by the child. It is the algebraic 5 and 5 and 1 compared with 1 and (5 and 5). The 5 must be added to the 5 before the child perceives that the addition of the 1 results in the 11. It is after this that he must perceive that the addition of 1 and 5, or of 5 and 1, is the addition of 6. This, too, is more readily seen by the latter plan, where it has not been necessary to overemphasize the addition of the 5 to the 5. Under this system of grouping, which it may be convenient to designate as the Colburn, there are nine groups containing from nine to one elementary facts.

2. *Why the Grouping Under Each Successive Digit of the Sums Which Have It for Their Result is Not the Psychological System of Grouping.*

This is of course the Grube system of grouping, limited to a certain class of fundamental facts. For thirty years or more it has been gradually superseding the Colburn plan, until to-day there are few if any elementary arithmetics that do not follow it. The element common to the sums composing one of its groups is their result, and that alone, unless, as is not done in the Grube method of teaching, the result is considered in its relation to that of the

group immediately below. In this latter case, it may be regarded as one greater for each sum than the result for the corresponding sum in the lower group,—a common element readily enough perceived after it has been obtained, but hardly effective, in that, for example, 4 and 2 are known to be 6 because 5 and 1 are 6. The child could only recognize an unknown sum as belonging to the 6 group through the fact that another sum, one of whose addends was identical with one addend of the unknown, and the other less by one than the other addend of the unknown, had 5 for its result,—a generalization which once mastered would be applicable to all groups, but whose mastery is at least sufficiently difficult to have prevented any attempt to utilize it by the disciples of Grube. Under the Grube order each elementary sum has been taught as a fact separate and distinct from other elementary sums, —except in so far as with each sum has been taught its inversion, which practice with that of simultaneously teaching the corresponding elementary differences, products and quotients, need not now be considered. As a grouping is possible in which the mastery of one elementary sum in a group will insure the ready mastery of all, the Grube order is obviously unpsychological.

It has been continually urged in its favor that it insures the teaching of the "simpler" facts before that of the "more complex." By it, it is claimed the children are not confused by the introduction of the more difficult numbers, before the easier ones are mastered. Unfortunately for this argument, however, the great majority of children on entering school, are already familiar with the names of all the digits. In fact many a teacher of the Grube system has found herself forced into the absurd position of wishing that her children knew less. But granting that as far as number is concerned, the mind of the beginner is a *tabla rasa,* it by no means follows that by teaching the sums as isolated facts, the Grube order presents first those that are the "easiest" for the child. No fact has

been better demonstrated by psychologists than that the objectively simple may be subjectively complex, and the objectively complex, the subjectively simple. It has not, for example, been satisfactorily shown, that the association 5 and 4 is more difficult for the child than is that of 2 and 3. Even aside from this, objectively considered, it would be hard to express in mathematical terms the difference in "simplicity" between 4 and 2 and 8 and 1. And yet, it is only in so far as through it the beginner may be first taught those individual facts, which he can most readily master, that the Grube order is in any respect an improvement upon the most arbitrary order of teaching the elementary sums which will include them all, especially as it divides them into seventeen groups, where the Colburn groups them in nine.

3. *Why the Grouping Together of the Sums Obtained by the Successive Addition of Each Digit, First to Itself and Then to Each Inferior Digit, is Psychological.*

There remains to be considered the system in whose successive groups the constant term is the digit added. While it has never been used so generally as the two already discussed, it has occasionally found its champions, as, for example, when in the early sixties, Felter and Eaton used it in their elementary arithmetics. In it, the element common to the facts included in a given group is the number to be added combined with the fact that the result is always one greater than the corresponding result in the group immediately below. Unlike that just considered, this common element is highly effective. For example, the pupil who is once led to see that 5 and 3 are 8 because knowing that $5 + 2 = 7$, he will know that $5 + 2 + 1 = 8$, will readily come to derive for himself the result of every other elementary sum involving the addition of 3.[19] That the element thus effective was readily perceivable by the great majority of the pupils in the first school grade of one representative locality, will be presently demonstrated.

Meanwhile, being effective,—as the common element in the Grube group is not,—being, if not more effective, at least far more readily perceivable, than that of the Colburn series, it is plain that the system of which it is the basis, with its nine groups, is the psychological system.

The pupil who has once made the simple generalization that one added to a number always results in the number next above it in the scale, will readily derive for himself with the aid of proper instruction—which in its primal sense means, after all, the right arrangement of subject matter—group by group, all of the elementary sums. The thorough mastery of one group insures the ready derivation of the next. Knowing all the elementary sums obtainable by the addition of four, he will readily derive and for the most part, quite abstractly,—all of those obtained by the addition of 4 and 1 or 5.

As this is the case with all the groups and the derivation of each group depends upon the thorough mastery of that immediately preceding, it follows that the system under discussion, being psychological throughout as compared with either of the others, it is unnecessary to use groups from more than one system.

(IV) The Determination of the Psychological Order of the Groups Selected as a Result of This Comparison.

One group being dependent for its derivation upon the thorough mastery of the group immediately below, the psychological order of the groups within the system is naturally determined: First, the elementary sums formed by the addition of 1; second, those formed by the addition of 2, etc., to those formed by the addition of 9.

(V) The Determination of the Psychological Order for the Facts Within Each Group.

The elementary sums are not only to be independently derived by the pupils, but so thoroughly memorized that the result will be instantly called to mind, when the

addends are seen or heard. This ready recall can be consequent only upon repetition,—a repetition either abstract and mechanical, or varied and concrete,—of the isolated fact or of the facts in relation to each other. For example, 4 and 2 are 6, once derived, can be memorized either by repeating concrete applications—such as 4 cats and 2 cats, or 4 cents and 2 cents, or by a mechanical repetition of the abstract $4 + 2 = 6$. Neither plan is wholly satisfactory. The first is uneconomical, in that too much time is taken in getting the necessary amount of repetition. Though properly used, it appeals to the child's interest, and though applying the facts, it is a necessary form of the number drill, it is not the form of repetition which will result in the memorizing of the facts with a minimum expenditure of time and effort. The second is economical until the attention of the pupil is lost through his lack of interest and the fatigue arising from the unvaried repetition. It can be occasionally used, to fix in the pupil's memory some individual fact, but is manifestly unfitted for the regular drill upon the facts of a given group.

Again the 4 and 2 are 6 can be memorized in an abstract series where it is associated with other facts,—as in the "addition table." In the case of the tables as usually arranged, the pupil mechanically commits to memory a series of facts,—which he may or may not have derived for himself,—in such a way that when a fact is demanded out of its usual order, it often can be recalled only through a running over of that part of the table immediately preceding it, to the end that a false and unnecessary association may be utilized. Nevertheless the table has always been popular with the pupils, and much of the disrepute into which it has fallen among the "more advanced" teachers, is due to their naturally but unnecessarily associating it with the irrational instruction of which it so often formed a part, and their erroneously assuming that the child finds the monotonous repetition as unpleasant as does the adult.

It is this very rhythmical, sing-song movement that gives pleasure to the child; it is this combined with the joy of achievement that makes him love to count.[21] If the fundamental sums could be so arranged that the association formed by repeating them in succession would be an essential association of result with its addends, rather than the false one of sum with sum, the repetition of these facts in tables would be the psychological way of memorizing them, because it would be the readiest way, and the way most pleasant to the child. Now it is just such an arrangement that results from the sub-grouping of the sums formed by the successive addition of a given digit, into those in which it is successively added beginning with itself, and those in which it is successively added beginning with each inferior digit,—that is, the arrangement which is based upon counting by 2's from 2 and from 1; by 3's, from 3, from 1 and from 2, etc.[20] Of course this drill includes not only the elementary sums not inversions, but all the fundamental addition facts. Two and 4, though identical in its result with 4 and 2, and 14 and 2, though but 4 and 2 and 10, must be as readily recalled as the 4 and 2 itself.

Knowing as the children do, that they are counting say by 4's from 2, as they repeat "6, 10, 14, 18," while the teacher interjects "and 4?" they are forming the association 6 and 4 are 10, 10 and 4 are 14, etc., in such a way that the series once mastered, all of its individual facts are mastered. The drill is therefore economical. While abstract it is varied,—that is, it is not the mere repetition of one fact again and again. It is interesting to the children from their love of counting, due not alone to its pleasurable rhythm, but as they master successive series, to the joy of achievement as well—the satisfaction arising from the knowledge that they can count by 3's and 4's instead of merely by 1's or by 2's.[21] *Where a digit is successively added to itself, the way is opened for the ready derivation of the fundamental products and quo-*

tients.[22] *Finally, the only numerical knowledge which the great majority of children have in common on entering school,—the ability to more or less mechanically count by 1's, within narrow limits,—is utilized from the very start. It is with counting by 1's that the work begins.*

It is of course admitted that the counting is "mechanical memory work" but what of that, if, though abstract, it is interesting to the child, and, through the very fact that it *is* mechanical the readiest way of fixing firmly in his mind *facts that he has already intelligently derived.*

This sub-grouping, then, of the psychological groups is psychological, since the facts are so arranged that as each group is derived, it may be memorized by the child with a maximum of readiness and interest.

So far, it is only those of the fundamental sums which are less than 24, that have been psychologically grouped. There remains a number of groups no less fundamental, whose psychological order must be determined. The sums obtained by the successive addition of 10's from 10; the sums obtained by the addition of the various digits to the multiples of 10 thus derived; the sums obtained by the additions of 1 to every number from 24 to 99,—necessary to intelligent counting to 100,—while so related to each other that they must be taught in the order in which they were just named, should have their place in the general psychological order as certainly established, as that of the sums already discussed. To do this it is necessary to assume, what later will be convincingly proven,—that it is as easy for the majority of children to add 10's as to add apples, and far easier to add a digit to 30 or 40, or 1 to 25 or 48, than to add 2 to 6 or 9. This being the case, the psychological order for these groups of fundamental sums is immediately after the 1 sums from 3 and 1 to 9 and 1, and—on account of the difficult terms—11 and 12—before the 1 sums from 10 and 1 to 19 and 1. Thus the decimal concept is formed from the start, and a large number of fundamental facts mastered through the ready

mastery of a common element necessary to the comprehension of the decimal system.

(VI) The General Psychological Order of Teaching the Fundamental Sums, as Thus Determined on A Priori Grounds.

1. Intelligent counting by 1's to 10; the "one sums" and their inversions.

2. Intelligent counting by 10's to 90; the "ten multiples" and the "ten sums." Place value to 10's.

3. Intelligent counting both by the addition of the successive digits, to the 10 multiples and by 1's—(1) from 20 to 100; (2) from 10 to 20.

4. Intelligent counting first to 12 and then to 24, by 2's, 3's, 4's, 5's, etc., successively—(1) from 2, 3, 4, 5, etc., respectively; *e. g.* "4, 8, 12," or "2, 4, 6, 8, 10, 12." (2) Successively from all other digits less than the one by which the counting is being done; *e. g.* "1, 5, 9, 13; 2, 6, 10; 3, 7, 11." All the elementary sums formed by such successive addition of each digit, being with their inversions, thoroughly mastered, before the beginning of the drill on counting by the digit next succeeding.

(VII) The Determination of the Psychological Order of Teaching the Fundamental Differences.

In determining the psychological order of teaching the fundamental differences, it is necessary to consider not only the system of grouping in which they can be most readily mastered by the individuals taught, as a series distinct from all other facts, but also to fix the relation which their teaching should hold, to that of the fundamental sums—it being assumed for the present that the teaching of the fundamental products and quotients should follow after, except those whose derivation is incidental to elementary addition.

Should the fundamental differences be taught at all? If so, should they be taught contemporaneously with the fundamental sums, or after the fundamental sums have been thoroughly mastered? Whether they should be taught as distinct from the elementary sums, or in other words, whether subtraction should be taught other than as a particular form of addition, is a question which has not as yet been experimentally settled. *A priori*, there is little doubt that a difference can be just as accurately and as quickly obtained by considering its units, tens, etc., as the numbers to be added to the corresponding digits of the subtrahend in order that they shall equal those of the minuend, as by taking them to be the direct result of the successive subtraction of the corresponding digits of the subtrahend from those of the minuend. But while as a mechanical operation, the former may be as satisfactory as the latter, it is more difficult of application. For example, an individual who wishes to determine the remainder which will result from taking 9 from 17, must know that it is the number which added to 9 will give 17. This he can learn either unintelligently or intelligently; he can either blindly accept the fact that the difference is the number which added to the subtrahend will give the minuend, or accept it because he has made the generalization that the minuend being the sum of the subtrahend and some other number, if the subtrahend is taken away the other number must remain. As intelligent operation is here assumed, this generalization becomes a necessary condition to subtraction "by addition." But an individual who has made it can derive at will all of the fundamental differences. The customary drill in the rapid subtraction of small numbers, would therefore be coincident with drill upon the fundamental differences. In view of this fact, it is probably better to consider the making of the generalization, where it can be made, as the psychological method of deriving them, rather than as a necessary step in a mode of subtraction, which with them once mastered, presents no advantages over that in general use.

If the fundamental differences are to be taught, another question arises—shall they be taught simultaneously with the fundamental sums, or after the fundamental sums in whole or in part have been thoroughly mastered. Against the former alternative is the fact, that—at least where number is taught to children on their first entering school,—many are so immature in their mental development, as to be sadly confused by the introduction of the "less facts" while they are striving to master the sums. In favor of it is the practice existing in many schools of teaching sums and differences, inversions and alternations, together—the only psychological grouping possible under the Grube system. For example, with 5 and 1 and 1 and 5 are taught 6 — 1 and 6 — 5, all four of the facts having the 5 and the 1 variously associated with the 6. But this is, after all, merely the objective derivation of each fundamental difference, aided by the generalization already discussed. It is manifestly far more psychological to thus derive no more of the differences than are necessary to the making of the generalization, by which, once made, all can be derived without the aid of objects. The postponement of this derivation until after the first groups of fundamental sums are thoroughly mastered, will give abundant opportunity for all who are capable of making the generalization to be led to make it, and at the same time will allow those not capable of making it, and therefore in the stage of mental development where confusion of sum and difference will be most puzzling,—to devote their undivided attention to the mastery of the fundamental sums. If after those failing to make the generalization had thus mastered the sums, they would be compelled to objectively derive the individual differences with no aid from the grouping,—especially, if the number thus failing was,—as it proved to be, a considerable fraction of the whole,—the teaching of the fundamental differences after the sums, might not be psychological. But aside from the possibility of hav-

ing such pupils wait for the study of subtraction until they are more mature and are able with their "brighter" companions, to derive the fundamental differences at will, or to dispense with them altogether,—the differences, like the sums, can be taught in psychological order, and, by the same simple mental process, one group after another readily derived.

Where pupils, once having mastered the fundamental sums, can be readily led to make and to apply the generalization, that whenever one of two numbers is subtracted from their sum, the other remains, so far as the mere derivation of the fundamental differences is concerned, the order may be purely arbitrary. But it is necessary, not only that a difference shall be derived but that its minuend and subtrahend shall be so firmly associated with it—that is, with the "difference," in the narrower sense,—that perception of the former shall instantaneously call up the latter. Here, as in the case of the fundamental sums, counting seems to furnish the most psychological form of the repetition without which this firm association would be impossible.

Where pupils who have mastered the fundamental sums are unable thus to derive the fundamental differences, the psychological order of teaching the latter can be shown by precisely the same line of reasoning,—to be essentially the same—though in inverse order,—as that in which the former should be taught. The pupils will first learn to count backwards by 1's—say from 24. This will, with proper instruction, give them the fundamental differences, resulting from the subtraction of 1. Those resulting from the subtraction of 2 will then be obtained, by the successive subtraction of 1,—thus $18 - 1 - 1$ or $18 - 2$; those resulting from the subtraction of 3, from the subtraction of 1 from the corresponding differences of the "2 group," etc.

The sub-grouping, as before, will be such as to make possible counting backwards by 2's from 20 and from 19,

by 3's from 21, 20, and 19, etc.,—the rhythmical repetition resulting in the readiest memorizing of the fundamental differences, however they may have been derived. As when a series has been once thoroughly mastered, it is far easier to learn to repeat it backwards, than it was to originally master it, the fundamental differences will be more readily mastered than were the fundamental sums, and at the same time, the fundamental sums will be more thoroughly mastered than before. Hence the grouping together of the differences obtained by the successive subtraction of each digit first from its multiple which is next above 20 and then from the corresponding number for each inferior digit, is psychological:

1. For those pupils who failing to make the generalization find in the mastery of the effective element common to successive groups the readiest method of deriving the fundamental differences.

2. For all pupils who through love of rhythm find counting the form of repetition by which they can most readily memorize the differences,—whatever the method by which they have been derived.

And this grouping is psychological, whether the fundamental differences are taught simultaneously with the fundamental sums, or after the fundamental sums have been thoroughly mastered.[24]

Granting then, that *a priori* the psychological order of teaching the fundamental sums and differences has been satisfactorily determined, it yet remains to be demonstrated that these *a priori* determinations are correct. In the schools which had among their pupils the hundred children whose numerical knowledge was tested on their first entering the lowest school grade this demonstration has been attempted,[23] and the results of six months' work subjected to a rigid and impartial test. While only a wide adoption of the order recommended, in schools in which the conditions greatly vary, will make possible a series of general tests, which will be universally accepted

as demonstrative,—the results so far attained at least serve to indicate that such adoption would be justifiable, and that such experimental demonstration is not unlikely to prove the determinations to be indisputably correct.

Since so far at least as economical derivation is concerned, the results of teaching in psychological order are dependent upon the employment of psychological method, the report of the test will be preceded by a discussion of method, whose determinations like those concerning psychological order, the results of the test will tend to confirm.

CHAPTER III.—THE A PRIORI DETERMINATION OF THE PSYCHOLOGICAL METHOD OF TEACHING THE FUNDAMENTAL SUMS AND DIFFERENCES.

I. A DEFINITION OF PSYCHOLOGICAL METHOD.

That method of teaching a particular branch of study is psychological, which while insuring the readiest mastery of its subject-matter by the individual taught, gives a maximum of mental training along the lines of development for the furtherance of which the mastery of that subject-matter is peculiarly adapted. That no one method will be psychological for all individuals must of course be admitted. Since, however, most individuals are taught *en masse,* the method which *in the case of the great majority* meets the conditions just specified, is the psychological method *for the school.* In the absence of any satisfactory demonstration of the common assumption that all special mind training is general mind training, it is safer to judge what training should result from the study of a particular branch, both in the light of that assumption, and in that of its negation. Every subject involves in its mastery the predominance of some particular mental activity. All other activities of the mind are more or less involved in it as they are more or less involved in every mental state, but one predominates in the study as one predominates in any phenomenon of mind. Thus imagination is the ruling activity in the study of descriptive history or geography, and representation in the use of the copy-book. If all special training is general training, it follows that so far as the school curriculum is concerned, the general training—say of the imagination—should be given through those branches in whose mastery

it is the predominant activity. Otherwise much time will be wasted in overemphasizing the importance of some mental activity, which is only incidentally involved in the study of a given branch. If all special training is not general training, it follows that the training of the imagination should be limited so far as it is not purely incidental, to those fields of activity in which it will be naturally exercised, or to those which most closely resemble them. In either case the method of teaching a particular branch should be such as to train the mental activity necessary to its mastery. Other activities should be incidentally trained, but the teaching, in addition to mastery of subject-matter, should involve training concentrated upon the activity which predominates in the science or art which is being taught. Pupils who are mastering descriptive history, should reason about the facts with which they deal, but as the main object of their study is to make those facts real, the method of instruction should be planned to train the imagination, rather than the reason. In other words the intensive study of history is premature wherever it precedes or is taking the place of the realistic presentation of important events.

II. CONDITIONS NECESSARY TO A PSYCHOLOGICAL METHOD OF TEACHING THE FUNDAMENTAL FACTS OF NUMBER.

The important place that mathematics occupies in the curriculum has always been justified, and must still be justified, if it is to be justified at all, by the fact that of all the so-called mental disciplines, it more directly and more fully than any other, involves the pure reasoning activity of the human mind. If the special training in pure reason, popularly supposed to be involved in its study, is a *general* training in pure reason, it is the duty of pedagogy to make good the popular belief. If not, only such mathematics should be taught as can be permanently

mastered, and in such a way that the pupils will be thoroughly trained in such mathematical reasoning as they are likely to exercise in everyday life.

In either case, such training as is given should be mainly a training of reason; in neither should it be mainly a training of representation or imagination. The imagination can be better and more economically trained in some study in which it is the dominant mental activity. The visualizing of lines and angles, and plane surfaces in varying perspective—all the trigonometrical training recommended by Herbart in his "A. B. C. of Sense Perception," may or may not be a necessary part of the training of the child's imagination. It certainly, however, is no necessary factor in the study of number, even though it be a preparation for the visualizing of the forms of geometry. Much less necessary is special training in the visualizing of complex columns and rows of figures. In the teaching of mathematics, including the teaching of the fundamental facts of number, psychological method is that which while insuring the readiest mastery of the facts by the individual, will give a maximum of training to the reasoning activity of that individual's mind. It should assuredly carry with it training of other mental powers, but just as assuredly such training should be incidental to the training of reason.

This discussion must therefore submit method to the following tests:

1. Is a given method better adapted than any other to insure the readiest derivation by the great majority of the pupils, of the various groups of fundamental facts?

2. Is it better adapted than any other to insure the readiest memorizing by the great majority of the pupils of the individual facts when once derived?

3. Does it involve a maximum of mental training, chiefly through the exercise of the reasoning activity?

The method which stands these tests can be safely characterized as psychological.

III. SPECIFICATION OF THE GENERAL METHODS OF TEACHING ELEMENTARY NUMBER.

However ambiguous the term "method" may be,—used now by Comenius to apply to his general analogy between nature and instruction, or by the pedagogic inventor of to-day to designate some useful but restricted device, no one will deny that there are certain general methods of instruction—call them what you will—based upon the various ways in which the human mind acquires knowledge. The subject-matter of arithmetic may be acquired inductively or deductively, synthetically or analytically, concretely or abstractly. It has been shown that to teach the number-facts in psychological order is for the most part to teach inductively, while it is evident that to so teach the fundamental sums is to teach synthetically. That this teaching is here psychological, no one will attempt to disprove who remembers that it has for its object the firm association in the mind of certain number-facts, leaving each philosopher to his own devices for filling the child mind with his number "concept," though being based none the less upon the fact that to the child number is an expression of aggregation,—a concept with which philosophers are beginning to agree. While it is perhaps equally evident that the psychological order of teaching the sums and differences involves a method which is largely abstract, the existing prejudice in favor of the objective method, the general assumption that the concrete basis admittedly necessary to the comprehension of the abstract, involves the objective teaching of each individual fact of number, makes highly desirable a thorough exposition of all that which goes to prove the abstract method psychological.

With this end in view, the various "methods" of deriving and memorizing the fundamental facts, may all be included in the following classification:

1. The Objective Method, in which each individual fact is derived by the direct use of objects.

2. The Memorial Method in which so far as possible each individual fact is derived by the summoning up of the memorial images of objects.

3. The Abstract Method by which so far as possible the facts of each group are derived by abstract reason from those of the group immediately preceding.

IV. A COMPARISON OF CERTAIN OF THESE METHODS WITH A VIEW TO THE DETERMINATION OF PSYCHOLOGICAL METHOD.

(I) The Pestalozzian or Grube Method, in Which Each Individual Fact is Derived with the Direct Aid of Objects.

1. *Does the Grube Method Insure the Readiest Derivation of the Fundamental Facts of Number?*

However necessary the use of objects to the formation of the number "concept," however essential that such use should result in the discrimination, abstraction and grouping which has number for its product, it by no means follows that *each fact of number* should be objectively taught.[25] In order that an individual who has formed adequate concepts of the numbers involved, shall fully comprehend a fundamental sum or difference, it is only necessary that he shall comprehend the simple and obvious fact of addition or subtraction, and perceive for himself the invariable result. The objective derivation of individual facts is justifiable, if in order to perceive that the result is invariable, in order to intelligently accept it as an abstract fact, it must first be perceived to be true in various concrete forms, and then the generalization made that being true of these, it is true of all,—that 3 cubes and 2 cubes being 5 cubes, 3 splints and 2 splints, 5 splints,

etc., 3 and 2 must invariably be 5. It is a serious mistake however, to assume that it is necessary to repeat this mental process in deriving *every* fundamental sum or difference. Sooner or later, the generalization is made that every fact of number, however derived, is universal in its application, that if 2 and 1 are invariably 3, and 3 and 2 invariably 5, 3 and 4, found to be 7 in one instance, whether by a derivation that is concrete or one that is abstract,—will be known to be 7, no matter what the concrete form of the addition may be. No one would dream of teaching 36 and 2 objectively, in order to lead pupils to perceive by the repeated addition of various objects that whatever things may be added, 36 and 2 will always equal 38. They already know that if through abstract addition, 36 and 2 be found to equal 38, the result will be invariable, whether the objects concerned are splints or trilobites. It remains for advocates of concrete number derivation to show that this is not equally true of 6 and 2. Generalization requires but few particulars. After the child has found that 2 and 1—3 with one set of objects, are still 3 though the objects be varied,—and being 3 though the objects be varied, are therefore 3 whatever the objects may be, he will not require many repetitions of this process of abstraction and generalization in the derivation of other facts, before he will assume that each fact which he derives will likewise apply to all things. In cases where the abstract fact seems to be derived by the summoning up of one that is concrete,—as, for example, where a pupil failing to give the abstract 3 and 5 can be led to recall the fact that 3 marbles and 5 marbles are 8 marbles, would seem to indicate that abstraction and generalization had not yet taken place, if it were not that the recall is often followed by the immediate generalization—3 and 5 are 8. It is not that 3 and 5 had not been generalized, but that not having been thoroughly memorized, the familiar concrete association helped in its recall.

As then, the use of objects is not necessary to intelligent generalization in the case of each fundamental fact, it having already been clearly shown in the discussion of psychological order, that the fundamental sums and differences can be more readily derived by a simple process of abstract reason, it necessarily follows that objects should be used only in general drill in number perception and in the occasional verification of facts already derived, except in so far as their use is necessary to the mastery of the effective element common to each group.[26] The Grube method, therefore, objectively deriving each individual fact, would seem at least as regards derivation, to be uneconomical and unpsychological.

2. *Does it Insure the Readiest Memorizing of the Facts When Once Derived?*

When a number-fact has been once derived, it must be memorized by repetition,—now as in the days of Richter, and as it must ever be, "the mother not only of study but also of education." Just so far as the Grube method limits its memory drill to repetition with actual objects or of concrete facts, it is unpsychological. There is a tendency among the friends of the "new education" to confuse the mechanical with the irrational. Word for word memorizing of the subject-matter of education is mechanical, but it is irrational not on account of its being thus mechanical, but first, because the words of few text-books are worth memorizing, and second, because thoughts are more certain of recall through their manifold association with other thoughts. If words are to be thoroughly memorized,—no matter how full the apperception of the thoughts which they express, the necessary repetition should be "mechanical." So with the abstract fact of number. Five is to be permanently associated with 3 and 2,—an association of symbols and not of thoughts. This association is assisted by the common association of

the 5 and the 3 and 2 with cubes, splints and geometrical forms, only through the repetition involved in the process. At least no individual is likely to recall a number-fact because of its association with the same series of objects with which all other number-facts have been associated. But such repetition is plainly uneconomical when compared with the mechanical repetition of the abstract fact, unless, to be sure, the former is, as is assumed, more interesting to the child, and so holds his attention as the latter can not. It is, however, at least very questionable, whether this assumption of interest will stand the test of impartial investigation. The use of a great variety of attractive objects interests the child, but in the objects, not in the fact which they are intended to exemplify, while the use of the same set of objects again and again, —no matter how finely polished the cubes or how brightly colored the splints,—is no stimulus to interest, especially as in the end it must come to be associated with failure to recall and sometimes with more or less merited rebuke. Such interest as is manifested in concrete repetition may after all be traceable to a love of counting combined with the joy of mastering a new fact or recalling one that was forgotten, supplemented in many cases by the sympathy, enthusiasm and dramatic ability of a Pestalozzian teacher, rather than by the inert concreteness of Pestalozzian matter.

There is possible, however, another form of concrete repetition in which a given number-fact is firmly associated with a particular concrete thing, which often seen, will repeatedly recall the fact to mind. $2+2$ are 4 may thus be associated with the legs of a chair, or 2 and 1 are 3 with the vertices of a triangle. While granting that the numerical association would often arise when the objects were seen, and that this incidental and natural repetition would firmly fix the facts in mind, it would be manifestly impossible to select a sufficient number of familiar objects adapted to this system of repetition.

A priori, then it would seem that any form of abstract repetition of a number-fact once it has been intelligently derived, so limited in time as not to become wearisome, will result in a readier memorizing than will repetition in the concrete. Especially will this be the case, where the repetition can be made rhythmical and, so, pleasing to the child.

The Grube method therefore can not be psychological in so far as it involves repeated objective illustration *in the memorizing* of fundamental facts. Its use of memorial images or of concrete associations to the same end, will be considered later on in this discussion.

3. *Does it Involve a Maximum of Mental Training, Chiefly Through the Exercise of the Reasoning Activity?*

The most enthusiastic advocates of objective operation are hardly likely to urge it as a means to mental training. They rather assume it as a necessary condition. The discrimination into separate wholes of objects so similar that they can be perceived as a united whole, however necessary to "number perception," is not a mode of training necessary to supplement the child's experience. If the similarity were to be sought out, it would be valuable training of an essential factor of the reasoning process, but this would be merely adding difficulty to the formation of the "number concept." This analysis of wholes into component wholes easily apparent, far from training sense perception, serves but to strengthen the every-day experience, which discriminates wholes without perceiving them in detail. But even were training in sense perception and in the discovery of concrete similarities involved in the teaching of the fundamental sums and differences, it should be purely incidental to the training of abstract reason. Only in systematic nature-study and the experimental study of the natural sciences should such training become a dominant factor.

The only remaining form of mental training involved in the objective method is exhaustive drill in generalization. It has already been demonstrated that continual repetition of this process is unnecessary in so far as it is intended to lead the child to comprehend that a fact true for one collection of objects will hold for all. Generalization, and, too often, generalization from insufficient particulars, is a natural activity of the human mind. 3 and 2—5 in the one case actually tested—will be 5 in all, unless it is proven to be 6. It is only so far as the objective method acts as a check upon this dangerous tendency that it can be useful as training in generalization. As the fact that 3 and 2 will always prove to be 5 only serves to confirm the usual assumption, training which looks toward a surer judgment should be reserved for studies which demonstrate that one Emerald Island does not make all islands green, or that a broken rule does not infallibly indicate a malicious child.

Except in so far as all mental activities are involved in each, this is the only mental training involved in the Grube method. It not only fails to train the abstract reason, but involves the very minimum of mental training which can result in the mastery of the fundamental sums and differences.

(II) The Memorial Method, in Which, so far as Possible, Each Individual Fact is Derived and Memorized by the Summoning up of Memorial Images or of Concrete Associations.

1. *Does the Memorial Method Insure the Readiest Derivation of the Fundamental Number-Facts?*

All that has been urged against the excessive use of objects in teaching the fundamental facts of number can be urged with more than equal force against such teaching as is dependent upon the summoning up of

memorial images. Indeed no one would seriously advocate the deriving of 7 as $3+4$ through the recall of the visual images of 3 objects and 4 objects and their combination into 7. The memorial method, however, has been very generally used in the memorizing of facts when once they have been objectively derived, or objectively applied, whether as the result of instruction or of numerical experience outside of the school.

2. Does it Insure the Readiest Memorizing of the Facts When Once Derived?

It has already been shown that the very general use of "number stories"—however valuable as a drill in language,—is unnecessary as a drill in the application of a generalized fact. If the child knows that 4 and 2 are 6, he knows that 4 oranges and 2 oranges are 6 oranges without ever having applied that particular abstract sum to the limited stock of objects which he has come to use in his number stories. Unnecessary so far as the sums and differences are concerned, as a means to application, the story even more than the use of actual objects, is uneconomical as a mode of repetition. If the child really visualizes as numbered groups the objects that he names—which will presently be disputed,—all that has been urged against objective repetition, is fully applicable, with the additional argument that the visualizing is unnecessary. If he merely visualizes unnumbered objects or associates the unfamiliar fact with the name of some familiar concrete thing, his repetition is plainly uneconomical. The argument that he will be more interested in the concrete has been already met. That the association of the unfamiliar with the familiar will result in its recall and so in its frequent repetition has also been shown to be improbable. It only remains to be demonstrated that even though it is assumed that the child visualizes the objects used as addends or as subtrahend and difference,

he can not visualize sum or minuend as a numbered whole.

It is a generally accepted fact that but five or six objects can be simultaneously perceived, if indeed it has been satisfactorily demonstrated that even this number can be perceived *simultaneously*. It is possible through training to perceive more than five or six objects *instantaneously,* but in all cases where this has been successfully accomplished, it seems to be as the result not only of a well-trained observation, but of an estimation so ready as in many cases to be quite unconscious. That is, the perception is in all probability successive and not simultaneous. Where the objects to be perceived have been so placed as to prevent so far as possible their separation into more or less definite groups, patient and persistent training has failed to increase the number of objects "simultaneously" perceivable.[27] Hence it is obvious that the individual, looking at the group of objects which for the first time he has formed by the addition of say 8 to 9, can have no numerical concept directly arising from his perception of the group as a whole. It is not until he perceives it as four 4's and 1, three 5's and 2, or some other combination with which he is familiar, or until he has counted the objects composing it, that he knows it to be 17. That is, he infers its equality with the group resulting from the addition of two other groups, by perceiving that it may be exactly separated into those groups, or by counting proves it to be equal to 16 and 1. The only visual image, then, that can arise in his mind in response to the oral or visual "17," is not that of one group containing seventeen objects, but of a group composed of at least three other groups—as 6 and 6 and 5.

While it has yet to be determined to what percentage of those versed in elementary number such a visual image does actually present itself, it is evident that in the case of larger numbers, the increasing complexity of an image —no group of which can be greater than 6,—precludes

its presentation when the number which it would visually represent is seen or named. Even, for example, so small a number as 57 could not be visually represented by fewer than nine groups of 6 each and one of 3. It is perhaps safe to say that under ordinary conditions, in the case of the great majority of individuals, no visual, tactile, auditory, or kinæsthetic image of a group of objects or sensations, rises in response to a number-name. And it may be asserted with still more certainty that no such reproduction is necessary to the number concept.[28]

There remains for consideration that form of the memorial method in which the concrete association is with the written number symbol. Figures are objects. Should the abstract fact after it has been intelligently derived be memorized through the repetition of its written form? So far, the discussion has concerned itself with the oral memorizing of the abstract fact, the desirability for which no one is likely to deny. But much of the addition and subtraction which the majority of individuals are called upon to perform is not oral but written. More than this, there are those who favor such drill in the visualizing of the written process, that long and difficult problems can be solved mentally. While the memorizing of the oral fact is all that is necessary to written addition or written subtraction, while either operation may be made a purely oral one by the oral repetition of the numbers written,—it is obvious that operation will be more rapid where a visual or a semi-visual association has been formed. In written subtraction the sight of the digits to be subtracted should at once call up the oral or the visual differences, without the intermediation of the oral digit names,—a fact that is equally true of addition not in column. To this end, then, concrete association is necessary. The pupil should be drilled upon saying or writing the result, when the numbers operated upon are written one above the other,—it mattering not whether the necessary kinæsthetic chain is recalled by its visual or its auditory

image. The algebraic form,—$1 + 3 = 4$ or $5 - 2 = 3$, so common in the written work of the school is no help to this written association. It is only justifiable in so far as it is desirable to familiarize the pupils with algebraic expression, and has in all probability resulted not from such justification, but as a survival of the old addition and subtraction tables scattered by the Grube method into individual facts.

A different form of concrete association—the "semi-visual"—is necessary to the highest degree of facility in addition in column. After the addition of the two digits at the top of the column, the numbers to be added are not seen together, nor are they successively pronounced. The addition is not of two numbers written one above the other, but of a spoken number or its auditory image, with a number written in column,—each oral result or its auditory image being added to the number next above or below the last number added. It is evident that to insure rapid addition in column, drill on neither the oral nor the written abstract facts alone, is all that is needed. The drill should consist of the addition of a written digit to one that has been spoken.

The memorial method, then, has to be utilized in the drill preliminary to rapid operation.

3. *Does it Involve a Maximum of Mental Training Chiefly Through the Exercise of the Reasoning Activity?*

It is at once apparent that teaching of number by the memorial method involves more mental training than does that which is limited to the objective method alone. Here the power of the mind to recall the concrete, especially the visualizing power, is more or less exercised. But whether or not such training is general training, but little of it is necessary in the teaching of number, while it can be made far more effective in the study of drawing, geography or history in which it is an essential factor. More

than likely the power to visualize unnumbered objects and geometrical forms is all the visualizing power desirable in mathematics. Whether this power is developed by the memorial method is to say the least very doubtful. The representation which it usually involves is in all likelihood, that of vague and indistinct wholes, only resulting in images clear and distinct in all their parts when visualizing is made a main end of the number study, as in the case of the mental solution of lengthy problems. It is very questionable whether this power to visualize a complex numerical operation is a useful accomplishment. Occasions are rare when ready written operation will not fully serve the same purpose. It is necessary to discriminate sharply between fundamental number fully comprehended, and other distinct ends with which it has been confounded. Even correlation does not demand that the economical teaching of one branch shall be sacrificed, and the main end of the consequent training minified, in order that something else more or less related to its subject-matter or more or less involved in the training shall be taught with it.

(III) The Abstract Method, in Which, so far as Possible, Each of the Facts in Each Group Are Derived by Abstract Reason from Those of the Group Immediately Preceding, and Largely Memorized by Abstract Counting.

1. *Conclusion that the Abstract Method Insures the Readiest Derivation of the Fundamental Number-Facts.*

As it is only through the use of this method that the psychological order of teaching number can be fully utilized in the derivation of the fundamental sums and differences, it is plainly the psychological method, wherever it is adapted to the majority of the children taught. Objects are used only in so far as they are necessary to the

comprehension of the effective element common to each group, to the occasional verification of a fact abstractly derived, and to general drill in number perception. The intelligent derivation of one fact in a group insures the ready derivation of all. Indeed with many pupils, the intelligent derivation of the facts in the first two or three groups insures the ready derivation of all the remaining facts. By no other method can the fundamental facts be so readily derived.

2. Conclusion That It Insures the Readiest Memorizing of the Facts When Once Derived.

That the memorizing of the fundamental sums and differences by forward and backward counting by 2's, 3's, 4's, etc., is economical, has been plainly demonstrated. It will become unpsychological only, if as the series multiply, the child who enjoys counting by 1's and by 2's should become weary of counting by 3's, 4's and 5's. This constitutes the main method of oral drill. It should of course be supplemented by the abstract written drill in which with the two addends or the minuend and subtrahend written, the pupil calls out or writes the proper result, and the semi-visual drill preliminary to addition in column, in which the pupil gives the sum of the number named by the teacher and that indicated by a pointer in a column containing all the digits.

By means of these drills, not only may the facts be readily and economically memorized, but if for the first year or so, they are made preliminary to all working of examples there need never be any slow operation. From the first, pupils should not be allowed to engage in operation involving facts which they can not give with the highest degree of facility. Examples should be devised made up wholly of those sums or differences upon which thorough drill has been given immediately before the addition or the subtraction is to be performed. Thus

operation being invariably rapid, rapid operation will become a habit—the degree of rapidity being dependent upon the reaction time of the individual, and finding its maximum at the point where rapidity threatens to result in nervousness.[29]

3. *Conclusion That It Involves a Maximum of Mental Training—Chiefly in the Exercise of the Reasoning Activity.*

The abstract method not only makes possible the readiest derivation and the most economical memorizing, but involves a maximum of mental training—chiefly in the exercise of the reasoning activity. If adapted to the individual taught, it is therefore not only the mathematical, but unquestionably the psychological method of teaching the fundamental sums and differences.

CHAPTER IV.—RESULTS OF A HALF YEAR'S TEACHING IN THE ORDER AND ACCORDING TO THE METHOD JUST DETERMINED.

I. CONDITIONS UNDER WHICH THE ATTEMPT WAS MADE.

These *a priori* determinations were put to the test in the same schools in which effort had been made to ascertain the numerical knowledge possessed by the child on entering school. On the whole, existing conditions were distinctly unfavorable to success. In two of the schools the teachers were new and with practically no experience in teaching young children, while a third, after being taught for about three weeks by a teacher in the early stages of typhoid fever, and for two months by a substitute, was then placed in charge of a young girl, well-trained, but who had never taught before. The number of pupils under each teacher varied from twenty-eight to fifty-four, and averaged forty-five. In two out of the seven rooms, the great majority of pupils were beginners; in the remainder, most of the children had entered during the last school year, while many had been in school for a still longer period. No special attention was given to beginners. On the contrary, according to a long-established, though unfortunate custom, far less time was devoted to those classes which largely consisted of children just entering school, than to those in which the great majority were hold-overs. More than this,—the same amount of time was not allowed to corresponding classes in the different schools, one class of fourteen, for example, being given on an average twenty minutes of oral drill each day, while another of twelve was given twenty minutes of similar instruction. The more advanced of the beginners were not taught with others who had just entered, but in

classes in which hold-overs largely predominated. As the schools did not open until September 5, and the classes were tested during the first two weeks in March, the results on the average represent the work of about one hundred and twenty days—which for half of those first entering school could not have meant an approximate total of thirty hours of actual instruction and for the remainder from forty to fifty hours. As it was believed that the experience of teachers during a short period, unrestricted by *a priori* suggestions as to method of procedure, would be the only stable basis for detailed directions in the future, they were given no special instructions other than that they should take up the elementary facts in the order just described, and in such a way that the pupils should so far as possible be led to perceive each common element.

In view of the variability in teaching power, in length of drill period and in manner of teaching, taken in connection with the inexperience of some of the teachers, the few hours spent in actual instruction in number, the absence of detailed directions, and finally the fact that even the few and simple directions that were given were for a time in some cases misunderstood,—the highly satisfactory progress of the work, from the very first, appears to justify the *a priori* determination of psychological order and method.

Before proceeding to discuss the searching and impartial test given at the close of the first half year, with its significant though of course inconclusive results, it will be profitable to consider the suggestive data incidentally furnished by the work that was subjected to it.

II. SOME OBSERVATIONS UPON FACTS WHICH IT INCIDENTALLY FURNISHED.

While no children were reported as having any serious difficulty with mechanical counting, it was soon discovered that probably from 10 per cent to 20 per cent of

the whole number of beginners would find it hard to master the element common to the facts of even the first group—although such mastery was necessary to intelligent counting.[30] The same children almost without exception also had trouble in their word-study,—in the association of the word and its symbol. Indeed at this point it may be well to state that among the children tested, it was a very exceptional thing to find those who were in more advanced reading classes than number-classes or *vice versa*,—only three being classified higher in number and two in reading. The "slowness," then, of these beginners, who with some "backward" hold-overs from the year before, composed the lowest number section of each first grade school, was plainly due to some condition common to both number-work and word-study,—a condition which could be no other than the complexity of the association required to memorize word-symbol or number-fact. Children whose associations had so far been largely confined to those uniting a name with a thing or one name with another, found it well nigh impossible to associate a name with a complex written symbol, or the "3," "and," "2," "are," and "5," into a fundamental sum. As they could readily associate a single letter with the corresponding sound, and a number with that adjacent to it in the number scale, the mechanical counting to 10 or 20 already begun on their entering school was readily mastered. But how were they to be led from that as a point of departure, to the derivation and memorizing of the essential facts? All that was necessary to the derivation of the first group was their comprehension of the fact that the addition of one to a number, results in the number next above it in the scale. But at the start, the pupils comprising these lower classes, utterly failed to comprehend it. Hence the effort was made to simplify the process of instruction. After the children could mechanically count with a high degree of readiness from 1 to 9, the teacher began to interject the expression "and one,"

between the terms of the series as given by the pupils in counting the objects as she added them one by one. Thus after they had become accustomed to the interruption, the "and one" gradually came to carry with it its proper meaning, and to be firmly associated with each number and that which immediately followed it. The second step in the process was to drill them upon the ready giving of the number that came next in the scale after any other which the teacher called out. Finally they were led to observe that the addition of one denoted by "and one" always resulted in that now familiar "next" number. Within six weeks all but six of the "slow" ones had mastered the facts of this group, together with the inversions. Under the old plan they would have been vainly struggling to remember the two or three "simpler" facts which they had counted out with the objects every day. During the first half year, even the remaining six yielded to treatment and mastered the nineteen facts with their inversions. A little less than one-fourth of the beginners, however, in the corresponding time, failed to associate 20 with two 10's, 30 with three 10's, etc. As they were led to think of the "ty" as a short way of saying "tens," and permitted to say "twoty," "threety," and "fivety," in all cases where the corresponding contractions did not seem to be sufficiently suggestive, and as those classes who to the last pupil mastered these facts with but little difficulty, were just as "slow" as those who to the last pupil failed, the true reason for their failure to master this and the two following groups appears to lie not so much in their inability to learn, as in the false assumption on the part of their teachers that having had difficulty with the numbers from 1 to 10, there was little use in attempting to teach them the "larger" and "more difficult" numbers from 20 to 100.[31]

While at the close of the first half year 34 per cent of the beginners had not yet succeeded in firmly memorizing all the fundamental sums resulting from the addition of

2, very few of them had failed to comprehend, that the addition of 2 was equivalent to the addition of 1 and 1, Great care, however, had to be taken to avoid plunging those who were more immature into inextricable mental confusion—the result being brought about rather through suggestion than by syllogism. For example, the series "4, 6, 8," etc., was taught as follows: The teacher placing two cubes in plain view of the class,—"How many cubes have I?" adding one and then another—"Two and 1? and 1?" putting the two new blocks together—"What *are* 2 and 2?" etc. Of course for a time some of these pupils answered "4" to the 2 and 2, not because they perceived the identity of the 2 with the 1 and 1, but because the "4" was the last answer that they had heard. It was not long, however, before almost unconsciously the perception came and they were able to apply the resulting principle in finding other sums.

Very persistent repetition was necessary in the case of these "slow pupils," to insure the thorough memorizing of the facts thus derived. Again and again did the teacher judge that the association had been firmly made, only to discover when she somewhat relaxed the drill that her faith had not been well founded. Had the counting proved dull and tiresome to the child, opponents of number teaching during the first year, would have found additional argument in the case of this lowest third of the beginners. On the contrary, the counting continued to be for all pupils, from first to last, a continual pleasure. They loved the rhythm, they gloried in conquering one series after another, as they glory in achievements in their play. The satisfaction with which they counted by 5's or 6's, and the contempt but half assumed with which they who were thus advanced, would rush through 1's and 2's, settled once and for all the fear that the interest shown in counting by ones might not continue in the higher series.

Although after a careful attempt had been made in every school, the pupils, including the great majority of

beginners, had experienced little difficulty in learning to count backwards—less difficulty of course than they would have had in learning an entirely new series,—when it came to drill on individual facts, the number who persistently confused the sums and differences, was, as had been anticipated—too considerable to be ignored. It was plainly unpsychological to have a child who found some difficulty in so simple a verbal association as that between a given number, "and" and the digit next succeeding, judge between such association and that formed between the same number, "less," and the digit immediately preceding it. For many the judgment was too complex. Some first giving the sum and seeing disapproval in the teacher's eye, would give the difference. Others of different temperament, or more immature, familiar with both facts but confused by the sudden demand for a judgment, would remain in a state of uncertainty, undoubtedly painful, until relieved by kindly suggestion or stunned by sharp rebuke.

While a fair majority of the beginners and a very considerable majority of the other pupils made the necessary discrimination without apparent difficulty, as has been already asserted there seemed to be no reason why the minority should find a serious obstacle to progress in an operation which was by no means a necessary condition to the more rapid progress of the more mature. As it still appeared highly probable that the former might be more able to judge between the two sets of facts later on in the school year, after many of the sums had been certainly mastered, the teaching of the elementary differences was abandoned in every school but one.

This action found additional justification from another consideration already brought out in the *a priori* discussion. Early in the school year, every class had been asked and had readily answered the following question: "If I had an apple and a peach, and took away the peach, what would I have left? If I had the apple and the peach and took away the apple? All but an inconsiderable

minority were equally successful in answering the following: "I have some fruit, an apple and a peach. If I take the apple away from the fruit, what will I have left?" etc. When numbers were substituted for apples and peaches in the first question, the answer was given quite as readily. When, however, the sum of the two numbers was substituted for the term "fruit" in the second question, a majority of the answers were confused,—it being of course quite natural for the "slower" children to give the sum—the number last named—in place of the remainder. Nevertheless, with the aid of concrete illustration and tactful questioning, even this difficulty was overcome, by so large a percentage of the pupils, that it seemed quite possible that with proper training, the great majority might be led to form the generalization, that when one of two numbers is taken from their sum, the remainder will always be the other. As this generalization is "effective," in that through it all fundamental differences can be readily derived, the steps necessary to it were from this time on made a part of the regular drill. The general test which will be presently described, having been taken too early in the year for the results of this postponement and preliminary drill to be statistically reported, it may be well to state that the experiment was highly successful. Half, for example, of the pupils in one advanced class at the close of the first half year gave with absolute correctness and with very considerable readiness any fundamental differences corresponding to fundamental sums which they had mastered. This informal test was as much of a surprise to the teacher as to the pupils, and so far as was known, involved no fact which the class had previously derived. A majority of the pupils still failing, however, to master the generalization, it was found necessary for them to derive the differences as they had derived the sums,—the backward counting proving as popular and successful a process as counting forwards.

The abstract derivation of each new sum, involving as it does, the oral addition of three numbers, was the beginning of a thorough preparation for rapid operation. Wherever the teachers strictly excluded all examples involving facts insufficiently mastered and persistently carried on the various forms of preliminary drill, the habit of rapid operation was formed by the pupils. While there is at present no way by which this can be satisfactorily demonstrated to those who have not visited the schools, it is none the less a fact that the pupils thus taught do not know what it is to add or to subtract slowly.[32]

III. NUMBER TEST GIVEN AT THE CLOSE OF THE FIRST HALF YEAR TO 237 FIRST GRADE PUPILS.

(I) Object of the Test.

The test was made to determine:

1. The percentage of the whole number of pupils as well as of the beginners, who had thoroughly mastered the facts of each series, as shown in the percentage of correct answers immediately given.

2. The nature of the deficiency existing among the remaining pupils, as shown by the percentage:

(1) Of correct answers deliberately given.

(2) Of wrong answers immediately corrected.

(3) Of correct answers immediately given and uncorrected.

(4) Of incorrect answers immediately given and corrected wrongly.

(5) Of incorrect answers deliberately given.

(6) Of failures to answer.

3. The comparative readiness with which the facts were given, as shown by the average time it took the pupils of each class to give a fact in each series,—deduction being made for all deliberations.

That is, the tests sought to discover how thoroughly

the fundamental sums had been mastered and how readily they could be called to mind and expressed.

Immediate answers deliberately given, indicated a mastery, mechanical to a higher or lesser degree, as shown in the average time it took to give a fact in each group. The comparison of the time thus recorded with the records resulting from future tests given under the same general conditions, will afford the measure of the gain or loss in mechanical facility.[33]

Correct answers deliberately given indicate the power to rationally derive an unknown or forgotten fact from one that is known. Wrong answers immediately corrected are more likely the result of nervous strain due to the test, or of psychological or physiological conditions peculiar to an individual, than of ignorance of the fact required. Incorrect answers immediately given and uncorrected while not necessarily indicative of individual ignorance, make collectively a fair index of general inaccuracy. While incorrect answers immediately given and corrected wrongly, may be due to nervousness, as is probably the case with those which are corrected rightly, they are more likely, perhaps, to result from uncertain knowledge. Incorrect answers deliberately given, probably for the most part result from some error in the process of derivation or from the basing of that process upon an incorrect sum or difference, although they may originate in an effort to supply by a guess a forgotten fact that can not be rationally derived. Failures to answer, though frequently due to embarrassment, are here assumed to be the result of ignorance.

(II) Description of Test and of Precautions Taken to Insure Trustworthy Results.

Both in the classification just given, as well as in the method of conducting the test, every effort was made to prevent the more or less unconscious modification of con-

ditions, so often due to the conducting of an investigation by those interested in its results. Errors due to partiality in questioning,—that is, to the teacher's consciously or unconsciously asking the easier facts of a series from the less proficient pupils and the more difficult from those who are brighter, were avoided by the adoption of a fixed order for the facts of each series. This order was purely arbitrary, and included all sums in each series, or their inversions. In order that the test might measure the actual progress of the schools, the facts in each series were given together. Test was also made of the facts miscellaneously grouped.

The lack of uniformity in the rate of questioning, destructive to the value of the time tests, was reduced to a minimum by all facts being demanded by the same individual in as quick succession as distinctness of enunciation would allow, and for a period so limited as to practically eliminate variations due to fatigue. Whenever a pupil hesitated, deduction was made in the time—all answers appreciably longer in coming than the others being counted hesitations. In order to insure accuracy and to prevent interference with the recording of results, the time was kept by an assistant. The teacher took no part in the test except to note the names of the pupils that failed, and the probable explanation of their failure, where explanation could be given. The use of an exceedingly simple system of recording made it possible for the individual who did the questioning to note each failure in the proper class without loss of time,—his pencil being ever ready to mark a tally under "immediate answers," while "deliberations" afforded time for the necessary record.

All children present in the schools on the day when the test was made,—with the exception of one who had just entered school,—were subjected to it—237 in all. Of these but 68 were beginners who had been included in the original 100 tested, although many of the others had entered school since the beginning of the

school year. Each child was tested on one fact in every group which his class was thought to have mastered. The total number of facts demanded from the whole number of children was 1,838.

(III) Analysis of the Results.

1. *More Facts Were Mastered by the Pupils Than in Any Corresponding Period in Previous Years.*

Though the test was severe, its results fully justified the order and methods that had been followed in the work of the first half year.

In the past it was only the most advanced classes that had mastered the twenty-five elementary sums and the twenty-six elementary differences, involved in the study of the numbers from 1 to 10,—fifty-one facts in all, together with their inversions and alternations. A few products and quotients and the differences resulting from the subtraction of each digit from itself are not included in this comparison,—their mastery having been incidental to both plans of work.

Under the new system, 16 per cent of all the pupils and 15 per cent of the beginners had mastered 112 fundamental sums with their inversions—over twice as many facts as any class had previously mastered in the same time, and in addition could count by 10's to 100, could instantaneously add any number from 1 to 10 to any multiple of 10 below 100, could add 1 to any number from 1 to 100, and in many cases could immediately derive all facts in subtraction corresponding to the sums which they had mastered. Twenty-seven per cent of all pupils and 21 per cent of the beginners, were as far advanced as those just reported, except that they had mastered but ninety-nine fundamental sums, and that it was a smaller percentage that could readily derive the differences. Re-

spectively 39 per cent and 31 per cent had mastered 85 facts; 52 per cent and 34 per cent, 70 facts; 56 per cent and 41 per cent, 54 facts; and 79 per cent and 66 per cent had mastered 37 facts. Seventy-two per cent of all pupils and 52 per cent of the beginners could add 1 to any number from 1 to 100; 76 per cent and 63 per cent respectively could add any digit to any multiple of 10 below 100, and 68 per cent and 44 per cent could similarly add 10. Eighty-five per cent and 78 per cent respectively could count intelligently by 10's to 100, and all could add 1 to any number from 1 to 20. No facts have been included in these figures upon which the pupils were not thoroughly tested. The majority of the classes, at the time the test was made, had been more or less thoroughly drilled on the series next above the highest upon which they were tested, while several classes had begun the systematic derivation of the fundamental differences.

It should be noted that the contrast between the knowledge possessed by the most advanced classes in this and former years, was no greater than that existing between the attainments of the slowest groups. In the past it was a source of gratification to the teacher if the unfortunates of which these groups were composed had mastered more or less thoroughly the first eight or ten facts. Under the new plan no group had mastered less than nineteen sums, while a large proportion of "slow" ones had mastered the multiples of 10, could add the digits to them, and could add 1 to any number below 100.

Such results, attained under unsatisfactory conditions, and in spite of the blunders ever involved in a new attempt, would seem to indicate that the plan thus tested, properly carried out, will make possible either the mastery of more number-facts during the first school years, or what is perhaps more desirable,—the mastery of the customary number of facts with less time devoted to the study of number.

2. *The Test Showed a Remarkably High Percentage of Correct Answers.*

Notwithstanding the fact that every child present in the various schools,—with the exception already noted—were subjected to the test, 82.1 per cent of the 1,838 facts called for were given correctly. As in the 17.9 per cent of wrong answers are included those of feeble-minded, backward, or nervous children, and of pupils who had been but a few weeks in school—some of whom failed on most of the questions and a few on all—the true percentage of correct answers should be much higher. For example, in one school of forty pupils, four children made thirty-four out of the eighty mistakes. With these four eliminated, the percentage of correct answers for the school increases from 80.5 per cent to 87.2 per cent. In the same ratio the 82.1 per cent of correct answers for all schools would be increased to 89 per cent.

3. *The Pupils Displayed a High Degree of Certainty in Their Answers.*

Only 1.3 per cent of the answers were changed, when once they had been given—1 per cent of these being wrong ones that were corrected, and .3 per cent correct ones that were made wrong. In most cases these errors were the result of nervousness.

4. *The Answers Were Given with Highly Satisfactory Promptness.*

As the average time taken in asking the questions distinctly is included in the average number of seconds given in the following table, it has been thought best to give for comparison the corresponding averages resulting, when the individual who asked the questions in the test, asked them aloud to himself and himself gave the answers with maximum readiness. In each case this average,

which for convenience will be called the "standard," is the lowest resulting from three of these self-tests. The results are given for each of the three or four sub-groups into which the number of classes was divided.

	Stand.	1st Gr.	2d Gr.	3d Gr.	4th Gr.
I. 5 & 1, 1 & 2, etc.	2.25	2.26	2.62	2.99	2.59
II. 6 tens, 2 tens, etc.	2.20	2.15	2.18	2.18	2.59
III. 60 & 10, 10 & 10, etc.	2.55	2.46	2.43	2.48	2.77
IV. 20 & 4, 40 & 3, etc. ...	2.48	2.48	2.60	3.13	5.50
V. 26 & 1, 52 & 1, etc. ...	2.60	2.58	2.75	2.89	
VI. 6 & 2, 2 & 2, etc.	2.50	2.50	2.85	2.96	
VII. 12 & 3, 3 & 3, etc.	2.55	3.33	3.43	3.61	
VIII. Miscellaneous Sums ..	2.40	3.13	3.39	2.50	
IX. Miscellaneous Inversions	2.45	3.38	3.94	3.75	
X. 12 & 4, 4 & 4, etc.	2.50	3.48	4.04	3.25	
XI. 10 & 5, 5 & 5, etc.	2.50	3.02	3.56	4.37	
XII. 12 & 6, 6 & 6, etc.	2.50	4.28	4.38	4.12	
XIII. 14 & 7, 7 & 7, etc.	2.50	3.91	3.57		

It will be noted that in most cases the variation is not great,—much of it, notwithstanding all precautions,—being in all probability due to failure to question at a uniform rate, although the number of seconds for the first group is uniformly lower than those of the second, and those of the second than those of the third. For the first six series, the answers are given with a high degree of readiness. From the seventh series on, there is a very marked decrease in the readiness with which the answers are given, due to the fact that the pupils in most of the classes had been given less drill upon the higher sums.

The averages for series VIII, in which the "2" and the "3 sums" are mingled, are higher than those of series VI, but lower than those of series VII,—the indication being that the pupils could give the miscellaneous facts as readily as those of the groups. The fact that the average number of seconds in series IX,—where the facts of series VIII are inverted, is higher than that in series VIII, is probably due not to any less readiness in giving the ordinary inversions, but to the unfortunate fact that it contained such

inversions as 3 and 12, 2 and 16, 3 and 13, and 2 and 14, which are rarely called for either in addition or in drill.

On the whole, the facts were given with a highly satisfactory degree of readiness, especially when it is taken into account that the main reason for thus timing the pupils at a stage of their work when but comparative little drill had been given them in operation, was to make it possible to gauge the improvement that should result by the close of the year. It is altogether likely, however, that the variation due to oral questioning will in the end prevent a satisfactory comparison.[33]

5. *Demonstration of the Ability of the Pupils to Promptly Derive for Themselves Facts Upon Which They Had Been Insufficiently Drilled.*

This evidence of independence in derivation was one of the most significant of the results. When pupils hesitated, 3.2 per cent of the questions received no reply at all—owing to nervousness, dullness, or mental confusion; 4 per cent after deliberation were answered wrongly, either through their derivation being based upon an error, or through some mistake in the process of derivation, unless the children thus answering waited a moment or so to give a guess that might just as well have been given at once. Nine and four-tenths per cent, however, were answered correctly after more or less deliberation. If the facts had been taught in Grubean isolation such recall would have been impossible. The significance of the foregoing percentage is largely due to the fact that the pupils who thus hesitated, were for the most part those, who being behind the majority of their fellows, might be supposed to be less likely to abstractly derive the facts that they had forgotten.

It should be remarked in conclusion that throughout the test, the beginners showed as great accuracy, readiness and independence in giving such facts as they had

mastered, as did the pupils who had attended school before.

IV. SUMMARY OF THE CONCLUSIONS WHICH THE INQUIRY TENDS TO ESTABLISH.

At the close, then, of the first half year of work, the results attained all tend to demonstrate the correctness of the *a priori* determinations regarding psychological order and method. While both are psychological only in so far as they are adapted to the majority of individuals taught; while it is altogether likely that there are individuals to which they are not thus adapted; they have proved themselves psychological in the only schools where thus far they have been tried. Nor is it unreasonable to assume that based as they are upon psychological fact, should they be adopted by other schools, they may in course of time be as certainly confirmed by general experience and proven by scientific test, as they have been indicated by individual reason and justified by individual success. For the present, a single investigator tentatively asserts as the result of a patient and impartial examination of limited data, that the only numerical knowledge common to a majority of children on entering school is the ability to instantaneously perceive three or four objects as 3 or 4, and to count more or less mechanically to 10 or 12; that the majority of such children love to count, and as their work progresses—to count by 2's, 3's, etc., as well as by 1's; that this knowledge and this interest form the natural basis for the study of number; that there is no necessary antagonism between the logical and the psychological orders of teaching the fundamental sums and differences; that the order in which they can be most readily taught is that logical order which results from the successive addition or subtraction of each digit from all other digits,—in that each of the resulting groups contains an element which readily mastered in one or two of the facts to which

it is common, readily effects the derivation of all; that this method of derivation involves the maximum of abstract reason possible to the majority of children on first entering school and the minimum of concrete illustration necessary to intelligent work; that it is based upon the numerical knowledge possessed by beginners, insures from the start a maximum amount of the mental training peculiar to mathematics, and makes possible the use of abstract counting by 2's, 3's, etc.,—the readiest means of memorizing the sums and differences when once they are derived; that the order and the method which thus result in the readiest mastery and the highest training are mathematical, and, being adapted to the great majority of the pupils of even the first school grade, psychological; that since their proper use insures the mastery of more fundamental sums and differences in the usual time or of the usual number of sums and differences in less time, and gives the pupils the power to readily derive forgotten facts, they are economical; that rapid operation in greater or less degree, can be made invariable from the first, and so from the first can become a habit.

NOTES AND REFERENCES.

[1] "Vorstellungkreis der Berliner Kinder beim Eintritt in die Schule." Berlin Stadisches Jahrbuch. 1870. Pp. 59–77.

[2] "Contents of Children's Minds on Entering School," by G. Stanley Hall, Ph. D., LL. D.

[3] *Ibid.*, p. 12. Dr. Hall here refers to "Der Vorstellungskreis unserer sechsjahrigen Kleinen." Allg. Schul-Zeitung. Jena, 1879. P. 327 *et seq.* This article I have been unable to consult.

[4] "Contents of Children's Minds," p. 19.

[5] *Ibid.*, p. 34.

[6] *Ibid.*, p. 17.

[7] *Ibid.*, p. 49.

[8] This conclusion, while merely tentative in so far as it is based upon the foregoing results, is in accord both with common experience and the general trend of philosophical determination. The Committee of Fifteen asserts that "counting is the fundamental operation of arithmetic, and all other arithmetical operations simply devices for speed by using remembered countings instead of going through the detailed work again each time." ("Report of the Committee of Fifteen," published for the National Educational Association by the American Book Company. New York, Cincinnati, Chicago, 1895, p. 53.)

Believers in the Culture Epoch Theory can not but accept this assertion, on the ground that in the historical development of the race, counting preceded all other forms of numerical operation. ("The Teaching of Elementary Mathematics," D. E. Smith. Macmillan,1900, p. 49.) It finds its fullest justification, however, in Dr. D. E. Phillips' article on "Number and Its Application, Psychologically Considered," in "Pedagogical Seminary," Vol. V, No. 2, pp. 221–281. (J. H. Orpha, Worcester, Mass.) "The earliest and most rudimentary form of knowledge in the cognitive sense," says Dr. Phillips, "is a knowledge of a series of changes" (p. 228). This "series-idea," established by a multitude of successive and rhythmical sensations conveyed through the different senses, and embodying the rudiments of the number concept, finally becomes abstract—a general idea applicable to any series of successions. At this stage symbols of representation are necessary. Hence results counting, first the naming of the series without reference to objects of any kind, then the counting of objects independently of the order of number-names, and finally the counting of objects with the number-names applied in proper order.

"Counting is fundamental, and counting that is spontaneous, free from sensible observation, and from the strain of reason" (p. 238).

[9] Such records as investigators have made of the mental development of individual children fully confirm this statement. Perez reports the case of a child two and a half years old who could count to 12, but did not understand the expression "3 days" until "another and another and another" made it clear. To the same child a year was "many, many to-morrows." ("The First Three Years of Childhood," Bernard Perez. Edited and translated by Alice M. Christie. E. L. Kellogg & Co., 1894, p. 186.)

Similarly in a case reported by Preyer, by the twenty-ninth month numbering begins to be active to a noteworthy degree. "The child began, viz, on the 878th day, suddenly of his own accord entirely, to count with his nine-pins, putting them in a row, saying with each one, eins (one) eins! eins! eins! eins! Afterward saying, eins! noch eins (one more)! noch eins! noch eins! The process of adding is thus performed without the naming of the sums." ("The Development of the Intellect," W. Preyer. Appleton, New York, 1895, p. 172.)

[10] Among 453 children first entering the schools of Chester, Pa., in September, 1900, but 56 (*i. e.*, 12½ per cent) were found unable to count to 8 or 10.

[11] While Preyer's generalization that "the boy of four years counts objects with effort up to six" and that "numbers remain for a long time merely empty words" (Dev. of Int.," p. 271), may be based upon too few particulars, the latter statement at least is undoubtedly correct. (See Dr. Phillips' "No. and Its Ap.," p. 234.) Cases where "counting proceeds independently of the number-names" (Phillips, p. 234), as where a child of nineteen months counts "2, 3, 5, 6, 7, 8, 9" (Preyer), while probably characteristic of "at least nearly all children" when they first learn to count, are by no means common among the children whom I have tested on their first entering school—probably less than half a dozen in several hundred. The inference is plain; most children in all probability begin to count long before they reach school age.

[12] The fact that in counting objects, children often run the number series ahead, counting "seven" before the teacher has placed the seventh object with the five or six, should not be taken to indicate inability to number objects. The error is due rather to a lapse in attention, and is not so likely to occur when the pupils are themselves handling the objects which they count.

[13] By "intelligent" counting, I mean nothing more than the numbering of things successively perceived, by a parallel repetition of the accepted number series, accompanied by the realization that each number-name applies to one thing more than that which immediately precedes it in the scale.

Of course this realization of the inclusiveness of each number-name involves the discrimination, abstraction and grouping justly emphasized by Professor Dewey (McLennan and Dewey's "Psychology of

Number," Appleton, 1898, pp. 31 and 32), the number-name denoting not the one more, but the number of ones in the group of which the one more forms a part. But when children who have mastered the number series abstractly, apply it to objects *in order to determine "how many,"* it is under conditions which insure the presence of each factor in the intellectual process.

Professor Dewey asserts that in order "to promote the natural action of the mind in constructing number, the starting point should be not a single thing or an unmeasured whole, but a group of things or a measured whole." So far as the "single thing" is concerned, he might have gone a step farther. The starting point in counting not only *should* not be a single thing, but *can* not be so. Before an individual attempts to determine "how many," there must be in his mind the thought of the unified group that he is to number. Its unity need not be that of qualitative likeness. It may consist in the mere fact of possession or location, as when he numbers the Christmas presents that he has received, or the various objects on a table. Ignoring qualitative differences with ease, or failing to observe them, abstraction presents less difficulties to him than to the philosopher. He unhesitatingly counts things that the philosopher would tend to subgroup. But the thought of some constant unity such as the number of legs possessed by a horse, or of a constantly increasing unity such as the telegraph poles seen from the car window, always precedes his counting. There is never a "how many" without a supplementary "what?" the thought of which precedes it, and the "what" is never a *single thing*, but always a unified plurality.

Just as generalization, abstraction and grouping thus precede the counting of things, discrimination acccompanies it. As each thing, observed in its proper sequence, serves as a stimulus to the corresponding number-name, it must be clearly perceived as a one of right belonging to the group. If things can not be discriminated they can not be counted. It therefore follows that if children succeed in applying the number series to things at all, they are grouping, abstracting and discriminating like so many philosophers. Professor Dewey does not give too philosophical a meaning to children's counting (Phillips, p. 243); he merely describes in philosophical terms an experience which children necessarily have whenever they apply the number series to things, but of which they are—and may be permitted to remain—unconscious. Before they attempt to determine how many, they generalize; if they count things at all they discriminate them.

If he would go on to assert that the naming of each successive number as one thing more is added to the group already counted, carries with it a recollection and discrimination of the individual things already numbered and their regrouping with the additional thing into the new unity to which the number-name is applied, I should venture to dispute

it on what appears to me fully adequate grounds. (See p. 61 of present discussion.)

That there is a vague recollection of the mass of *number-names* that have gone before, made possible in the case of the larger numbers, by the decimal terminology, I believe to be true. When in counting we say "twenty-six," there is no clear recollection, visual or otherwise, of 26 things (see p. 62 of present discussion), but rather the abstract thought of 26 as 25 and 1, and the vague knowledge—perhaps not present in every instance, but always possible—that many successive additions of 1 were necessary before the first 1 was increased to 25. When the decimal system has been mastered, added to this, if not entirely displacing it, there may be the abstract recall of two tens and six, a concept fully adequate to the demands of every-day life, if not to mathematical philosophy.

A vague content may be given the number-name long before mechanical counting becomes intelligent. Indeed, Professor LeFevre asserts that from the first there is a fuller recall of number-names, and hence a more exact notion of number than I can venture to claim. On p. 23 of his "Number and Its Algebra" (Heath, 1896), he states that "the order being learned by rote, any word numeral by suggesting its definite place in the fixed series of words, recalls all those gone before; and from this comparison the mind conveys or receives an exact notion of the number of individuals in the group of objects numerically characterized by any such number-name."

[14] P. 249, "Number and Its Application."

[15] See pp. 171–176 in "Outlines of Psychology," translated by Dr. E. B. Titchener from the German of Professor Oswald Külpe. (The Macmillan Company, New York, 1895.)

[16] See early files of the "Pennsylvania School Journal" and other educational periodicals.

[17] See "Elements of the Grube Method," Levi Seeley, M. A., Ph. D. (E. L. Kellogg & Co., New York.)

[18] Occasionally the general order is varied, as where 2, 4 and 8 are analyzed before 3, 5 and 7, but it may be safely asserted that in almost all American schools the various combinations and separations of a particular number are exhausted before the teaching of any fact involving the numbers above it. A useful piece of work yet to be accomplished is the certain determination of the number of schools using each of the possible orders of presenting the fundamental number-facts and of the various general methods of teaching them.

[19] The naturalness of such derivation as well as its rationality is seen in the following abstract from Louise E. Hogan's "Study of a Child." (Harpers, 1898, New York, p. 168.) "To-day we heard him say 'six and three are nine; six and four are ten.' His aunt asked him how he knew it. He replied, 'I know that six and three are nine, and four

is one more than three, and ten is next to nine, so it *must* be so.'" (See Note 26.)

[20] A summary of methods given in David Eugene Smith's admirable book, "The Teaching of Elementary Mathematics" (Macmillan, 1900, pp. 94–97), leads me to infer that this plan has been long in use in some of the elementary schools of Germany. He refers to Knilling's "Zur Reform des Rechenunterrichtes" and "Die Naturgemasse Methode des Rechen-Unterrichts in der Deutschen Volksschule," and to Tanck's "Rechnen auf der Unterstufe," "Der Zahlenkreis von 1 bis 20," and "Betrachtungen uber das Zahlen." To these works I unfortunately have not had access.

[21] Some who read this statement may prefer to agree with Dr. Dewey, when, after asserting that "the concept two involves the act of putting together and holding together the two discriminated ones," he goes on to say, "It is this tension between opposites which is largely the basis of the childish delight in counting." ("Psychology of Number," p. 31.) It seems to me, however, that the delight is in mechanical counting—which children love without any thought of the fact that one and one are two.

While Dr. Phillips finds that the "passion for counting" objects, most common in children between the ages of seven and ten, had its beginning in a "passion for following the abstract series" in but 57 cases out of 131 (Ped. Sem., V, p. 252), I have failed to discover any children of school age who do not heartily enjoy the rhythmical flow of mechanical counting even when "sing-song" is prohibited.

[22] See Phillips, p. 265. "Counting is fundamental and even combinations furnish the first step, hence counting by 2's, 3's, 4's, etc., furnish the first steps in multiplication."

[23] Encouraged by the results of this first attempt at demonstration, during the past school year (1900–1901), I followed the same plan in teaching the elementary sums and differences to 1,200 children constituting the first grade in the public schools of Chester, Pa. While I shall from time to time refer to the results there attained, it is perhaps well at this point to make the general statement that this second attempt was even more successful than was the first.

[24] See Phillips, p. 265: "The difficulties of subtraction are greater and more lasting than those of addition, hence its introduction on beginning addition must at first be incidental. Care must be taken not to confuse the child by introducing too many processes at once." While the results of my own work show the wisdom of Dr. Phillips' caution (see present article, p. 48), I find that after pupils have thoroughly mastered the fundamental sums formed by the addition of two, they can take up the corresponding differences without any serious confusion of the two sets of facts.

[25] An early protest against the "Bitter End of the Object Method"

was made by Ida F. Foster in *Educational Review*, March, 1894. "It is well that a child should see that four straws and three straws are seven straws, before he learns to say that $4 + 3 = 7$, but it is not necessary that four splints and three splints should be set before his eye every time he is asked how many $4 + 3$ make." Surely someone, though unknown to me, must have long ago pointed out the equal absurdity of objectively deriving each new fact.

[26] Hensleigh Wedgwood, in his article on "The Foundation of Arithmetic" ("Mind," Vol. III, p. 572), declares that the Grube Method assumes with Mill that the number-facts are mastered only "by constant observation of the result when groups of actual objects are combined or separated, *i. e.*, by experience." Not so, he goes on to state, with *the derivation of numbers by mental association*—"from consciousness on the contemplation of our own thoughts, that what we mean by two is nothing else than the aggregation of $1 + 1$." When with the aid of objects, children have learned that $2 + 1$ are 3, and that $3 + 1$ are 4, they can be quite readily led without other aid than their own powers of generalization unconsciously exercised, to the consciousness of the fact that $8 + 1$ are 9. This statement I make not as a reasonable assumption, but as an assertion based upon the experience of 1,500 children.

[27] A prolonged effort which I made a year or so ago to train twenty young men and women to instantaneous recognition of the number 8, with counting and estimation both conscious and unconscious practically eliminated, through the use of slightly curved lines of equidistant and uniform black dots, resulted in their ready recognition of the particular 7's, 8's and 9's on which they had been drilled, but in no appreciable improvement when they were called upon to distinguish from each other, 8's, 7's and 9's arranged in curves slightly different from those used in their training. Since with irregular arrangement I found training in recognition quite easy, I infer that results such as those which have been achieved by Miss Aiken (see "Mind Training"—Catherine Aiken) are due to training in instantaneous and perhaps unconscious estimation rather than to simultaneous number perception.

[28] Children often distinctly visualize the smaller numbers when counting. Of this Sully gives us a good example—his child saying, "One, two, three, four—two dollies, a tin soldier and a shell," as he successively pointed to their images on his coverlid. (P. 484, "Studies of Children," Appleton, New York, 1896.)

To what limit we can appreciate the number of units involved in any written or spoken quantity is problematical. Dr. Conant thinks 100 somewhat high for most of us, and 1,000 a rather certain maximum. (P. 33, "The Number Concept," Macmillan, New York, 1896.) According to Kay, in his "Memory, What It Is and How to Improve It,"

(p. 320, Appleton, New York, 1895), "G. P. Bidder, before he was six years old, was accustomed to amuse himself by arranging peas, marbles and the like in rows and squares of different numbers, and by counting them over to ascertain the results of the combinations. Hence the figures did not present themselves to him merely as symbols, but they represented to his mind an equal number of definite objects. He never went beyond 100." As a result, however, "he could multiply a row of fifteen figures by another row of the same number and give the actual result in a few minutes without seeing or writing down a single figure." Such use of imaged objects by a mathematical prodigy—probably as a mnemonic system—furnishes no justification for the excessive use of splints and cubes in teaching the fundamental number-facts to normal children.

In the *Educational Review* for 1893 (p. 467), Adelia R. Hornbrook claims pedagogical value for "number forms," and suggests a system of instruction based upon imaged columns of the numbers written in convenient forms—a device which in many cases might be practicable, but which in most cases would be unnecessary, and hence uneconomical.

[29] Kay asserts ("Memory," p. 321) that if the fundamental sums are so fully associated together in the mind, "that afterwards, whenever the two forms occur, the third being their sum, will at once come up—one may sum up a whole column of figures almost at sight. This is best done by simply using the eye without naming the figures." That is, the column 3, 2, 1, 4, etc., should be read "5, 6, 10," without naming the $3 + 2$, the $5 + 1$, etc.

But, Kay goes on to say, he will never thoroughly learn the combination $9 + 8 = 17$, "if he immediately adds, and $6 = 23$, and $7 = 30$. The mind must dwell upon one set of figures, and thoroughly master it before proceeding to the next."

When the number-facts, as here suggested, are taught in groups formed by the successive addition of the same digit—as, for example, "2 and 2, 4; 4 and 2, 6; 6 and 2, 8," etc.—it is the introduction by the teacher of the "and 2" into the series, which thus causes the mind to dwell upon the individual facts, and so completes the necessary association. This introduction of the common element should therefore invariably supplement the simple counting, in order that six will come after four in the children's minds only when they are counting by two's, and after five when they are counting by one's. Then, as the result of the mechanical drill, 4 and 2 will always be 6, and not 5 or 7.

[30] Of the 453 beginners in Chester, 12 per cent after learning to count had difficulty in telling what number came next after each digit, and 15 per cent were slow to learn the "one sums" to ten.

[31] This inference has been borne out by my later experience.

[32] In order to achieve this end, it was found necessary to specially

prepare for the teachers the examples needed for proper drill and to insist upon their exclusive use in seat work as well as in class exercises. The semi-oral drill upon the two or three facts other than the "one sums" involved in each set of examples was invariably insisted upon before the pupils were permitted to add, and the corresponding written drill before they were allowed to subtract.

After examples were copied for desk-work, the pupils added or subtracted them simultaneously. Dawdling was thus eliminated, and time economized.

[33] No such comparison has been made. In a test necessarily oral on account of the main objects it sought to accomplish and the inability of beginners to write with sufficient ease, it was impracticable to allow each class an equal period of time, and to make comparison on the basis of the number of facts written or of examples solved—the only mode of procedure by which can be obtained the uniform results necessary to an exact comparison.